A TREATISE ON

ATTRACTIONS, LAPLACE'S FUNCTIONS,

AND THE

FIGURE OF THE EARTH.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

A TREATISE ON

ATTRACTIONS, LAPLACE'S FUNCTIONS,

AND THE

FIGURE OF THE EARTH.

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FOURTH EDITION.

London and New York:

MACMILLAN AND CO.

1871.

PREFACE TO THE FOURTH EDITION.

In this Treatise the methods are explained by which the form or figure of the Earth is determined, both by theory and measurement. Theory has been of eminent service in the whole investigation. It was the fluid-theory which gave the first conception of the form being a spheroid; and it is by assuming that it is a spheroid, thus suggested by the fluid-theory, and by no other means, that the geodetist can bring his measures to bear upon the problem; he decides which of all spheroids best suits the observations and measures, and by that indirect way shows that a spheroid is no doubt the mean form, of which the axes are about 7926.6 and 7899.7 miles.

The Treatise begins by the calculation of Attraction under various circumstances, taking the Law of Universal Gravitation as the basis. I first calculate the resultant force exerted on a point by an assemblage of particles endowed with this attracting power, and held together in the form of a sphere, homogeneous or heterogeneous; next of a spheroid; then of an irregular mass consisting of layers nearly spherical, thus approximating more and more to the case of the Earth. This investigation gives me the opportunity of introducing the remarkable analysis of Laplace, which I have endeavoured to put in a clear light, and to free from objections

which have been urged against it. The first part of the Treatise is closed with a Chapter in which is calculated the local effect on the amount and direction of gravity caused by irregular masses at the surface of the Earth, such as exist in table-lands, vast mountain regions like the Himalayas, and hollows filled by the ocean which is of less density than rock; and also wide-spread but slight deficiencies or excesses of matter in the crust below. All these are of importance in the problem which it is my ulterior design to solve, as they furnish the means of determining the effect of disturbing causes on the Pendulum and the Plumb-line, and of explaining anomalies which would otherwise be unaccountable.

The second part of the Treatise is occupied in calculating the Figure of the Earth, first upon the hypothesis of its being a fluid mass; secondly, from the data furnished by pendulum experiments, by the moon's motion, and by the precession of the equinoxes; and thirdly, on geodetical principles. For the first, it is shown that whatever its former history may have been, the form of the Earth's surface and of all its internal layers must, on the fluid-hypothesis, be oblate spheroids; and that the plumb-line must be everywhere a normal to its surface. The second method, in each of its three applications, is based on a demonstration by Professor Stokes of Clairaut's Theorem, assuming as data that the surface is one of equilibrium, is nearly spherical, and is spheroidal, quite independently of any considerations of the arrangement of the mass. The third or Geodetic method occupies the last Chapter of the Volume. The method of Bessel at present in use for this purpose is shown to be defective in one particular, and is corrected

so as to bring into the calculation the effect of Local Attraction. The degree of uncertainty which that disturbing element brings into the calculation of the Figure of the Earth is pointed out; and it is shown how, with great probability of a correct result, the ambiguity may be removed by a comparison of the three long arcs, the Anglo-Gallic, the Russian, and the Indian. I believe this is the first time that the mean figure has been calculated, the disturbing effect of local attraction being brought into the calculation throughout. A method is given for using arcs of longitude and azimuths, as well as arcs of latitude, in the investigation of the form of the Earth's surface. Some propositions are added on the sea-level, on mapping countries, on differences of local attraction in the stations of the Indian Arc, and remarks on the hypothesis of the original fluidity of the Earth.

In the course of the Work I have applied the results of pendulum experiments to test a theory which I have propounded (Art. 192), viz. that the variety we see in the surface of the Earth in mountains, plains, and ocean arises from the Earth having assumed its form from being originally in a fluid state, and having contracted unequally since solidification began, the contraction in mountainous regions having been least and in the oceanic parts greatest.

JOHN H. PRATT.

CALCUTTA,
November 8, 1871.

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ERRATA.

PAGE 25, lines 3 and 4: for $d\mu$ and $d\omega$ read $d\mu'$ and $d\omega'$.
,, 35, Example 3, μ should be a factor of the second term.
,, 36, line 2, *ab imo* for $2i-2$ read $2i-3$.
,, 133, line 15, for *at* read *out*.
,, 173, last line, (t_3) is left out.

ATTRACTIONS AND LAPLACE'S FUNCTIONS.

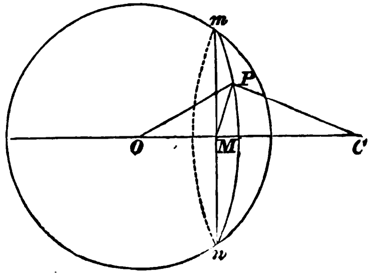
1. THE Law of Universal Gravitation teaches us, that every particle of matter in the universe attracts every other particle of matter, with a force varying directly as the mass of the attracting particle and inversely as the square of the distance between the attracted and the attracting particles. Taking this law as our basis of calculation, we shall investigate the amount of attraction exerted by spherical, spheroidal, and irregular nearly-spherical masses upon a particle, and apply our results in the second part of this Treatise to discover the Figure of the Earth. We shall also show how the attraction of irregular masses lying at the surface of the Earth may be estimated, in order afterwards to ascertain whether the irregularities of mountain-land and the ocean can have any effect on the calculation of this figure.

CHAPTER I.

ON THE ATTRACTION OF SPHERICAL AND SPHEROIDAL BODIES.

PROP. *To find the resultant attraction of an assemblage of particles constituting a homogeneous spherical shell of very small thickness upon a particle outside the shell: the law of attraction of the particles being that of the inverse square.*

2. Let O be the centre of the shell, P any particle of it, $OP = r$, dr the thickness, C the attracted particle, $\angle POC = \theta$; $mPMn$ a plane perpendicular to OC, ϕ the angle which the plane POC makes with the plane of the paper, $PC = y$.



The attraction of the whole shell evidently acts in CO.

Let OP revolve about O through a small angle $d\theta$ in the plane MOP; then $rd\theta$ is the space described by P. Again, let OPM revolve about OC through a small angle $d\phi$, then $r\sin\theta d\phi$ is the space described by P. And the thickness of the shell is dr. Hence the volume of the elementary portion of the shell thus formed at P equals $rd\theta \,.\, r\sin\theta d\phi \,.\, dr$ ultimately, since its sides are ultimately at right angles to each other.

Then, if the unit of attraction be so chosen, that it equals the attraction of the unit of mass at the unit of distance, the attraction of the elementary mass at P on C in the direction CP

$$=\frac{\rho r^2 \sin\theta\, dr d\theta d\phi}{y^2}, \ \rho \text{ the density of the shell};$$

$$\therefore \text{ attraction of } P \text{ on } C \text{ in } CO = \frac{\rho r^2 \sin\theta\, dr d\theta d\phi}{y^2}\,\frac{c - r\cos\theta}{y}.$$

We shall eliminate θ from this equation by means of

$$y^2 = c^2 + r^2 - 2cr\cos\theta,$$

$$\therefore \sin\theta \frac{d\theta}{dy} = \frac{y}{cr}, \quad c - r\cos\theta = \frac{y^2 + c^2 - r^2}{2c};$$

$$\therefore \text{ attraction of } P \text{ on } C \text{ in } CO = \frac{\rho r dr}{2c^2}\left(1 + \frac{c^2 - r^2}{y^2}\right) dy\, d\phi.$$

To obtain the attraction of all the particles of the shell we integrate this with respect to ϕ and y, the limits of ϕ being 0 and 2π, those of y being $c - r$ and $c + r$;

$$\therefore \text{ attraction of shell on } C = \frac{\rho r dr}{2c^2}\int_{c-r}^{c+r}\int_0^{2\pi}\left(1 + \frac{c^2 - r^2}{y^2}\right) dy\, d\phi$$

$$= \frac{\pi\rho r dr}{c^2}\int_{c-r}^{c+r}\left(1 + \frac{c^2 - r^2}{y^2}\right) dy = \frac{\pi\rho r dr}{c^2}(2r + 2r)$$

$$= \frac{4\pi\rho r^2 dr}{c^2} = \frac{\text{mass of shell}}{c^2}.$$

This result shows that the shell attracts the particle at C in the same manner as if the mass of the shell were condensed into its centre.

3. It follows also that a sphere, which is either homogeneous or consists of concentric spherical shells of uniform density but differing from one another in density, will attract the particle C in the same manner as if the whole mass were collected at its centre.

PROP. *To find the attraction of a homogeneous spherical shell of small thickness on a particle situated within it.*

4. We must proceed as in the last Proposition; but the limits of y are in this case $r-c$ and $r+c$; hence,

$$\text{attraction of shell} = \frac{\pi\rho r dr}{c^2}\int_{r-c}^{r+c}\left(1-\frac{r^2-c^2}{y^2}\right)dy$$

$$= \frac{\pi\rho r dr}{c^2}(2c-2c) = 0\,;$$

therefore the particle within the shell is equally attracted in every direction.

5. This result may easily be arrived at geometrically in the following manner. Through the attracted point suppose an elementary double cone to be drawn, cutting the shell in two places. The inclinations of the elementary portions of the shell, thus cut out, to the axis of the cone will be the same, the thickness the same, but the other two dimensions of the elements will each vary as the distance from the attracted point; and therefore the masses of the two opposite elements of the shell will vary directly as the square of the distance from that point, and consequently their attractions will be exactly equal, and being in opposite directions will not affect the point. The whole shell may be thus divided into pairs of elements attracting equally and in opposite directions, and therefore the whole shell has no effect in drawing the point in any one direction more than in another.

6. The results of these two Propositions are so simple and beautiful, that it is interesting to enquire whether these

properties belong exclusively or not to the law of the inverse square of the distance. To determine this is the object of the four following Propositions.

PROP. *To find the attraction of a homogeneous spherical shell on a particle without it; the law of attraction being represented by $\phi(y)$, y being the distance.*

7. The calculation is exactly analogous to that given above: we have only to alter the law of attraction. Then attraction on C in CO

$$=\frac{\pi\rho r dr}{c^2}\int_{c-r}^{c+r}(y^2+c^2-r^2)\,\phi(y)\,dy \text{ (integrated by parts)}$$

$$=\frac{\pi\rho r dr}{c^2}\left[(y^2+c^2-r^2)\int\phi(y)\,dy-2\int\{y\int\phi(y)\,dy\}\,dy\right]$$

$$=\frac{\pi\rho r dr}{c^2}\{(y^2+c^2-r^2)\,\phi_1(y)-2\psi(y)+\text{const.}\}\text{ suppose,}$$

$$=2\pi\rho r dr\left\{\frac{c+r}{c}\phi_1(c+r)-\frac{1}{c^2}\psi(c+r)-\frac{c-r}{c}\phi_1(c-r)+\frac{1}{c^2}\psi(c-r)\right\}$$

$$=2\pi\rho r dr\,\frac{d}{dc}\left\{\frac{\psi(c+r)-\psi(c-r)}{c}\right\},$$

this latter form being introduced merely as an analytical artifice to simplify the expression.

PROP. *To find the attraction of the shell on an internal particle, with the same law.*

8. The calculation is the same as in the last Article, except that the limits of y are $r-c$ and $r+c$:

$$\therefore \text{ attraction} = 2\pi\rho r dr\left\{\frac{r+c}{c}\phi_1(r+c)-\frac{1}{c^2}\psi(r+c)\right.$$
$$\left.+\frac{r-c}{c}\phi_1(r-c)+\frac{1}{c^2}\psi(r-c)\right\}$$

$$=2\pi\rho r dr\,\frac{d}{dc}\left\{\frac{\psi(r+c)-\psi(r-c)}{c}\right\}.$$

PROP. *To find what laws of attraction allow us to suppose a spherical shell condensed into its centre when attracting an external point.*

9. Let $\phi(r)$ be the law of force; then if c be the distance of the centre of the shell from the attracted point and r the radius of the shell, and

$$\psi(r) = \int \{r \int \phi(r)\, dr\}\, dr,$$

then the attraction of the shell

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}.$$

But if the shell be condensed into its centre, the attraction

$$= 4\pi\rho r^2\, dr\, \phi(c);$$

$$\therefore\ 2r\phi(c) = \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}$$

$$= 2\frac{d}{dc}\left(\frac{d\psi c}{dc}\frac{r}{c} + \frac{d^3\psi c}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \dots\right)$$

$$= 2r\phi(c) + 2\frac{d}{dc}\left(\frac{d^3\psi c}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \dots\right);$$

$$\therefore\ \frac{d}{dc}\left(\frac{1}{c}\frac{d^3\psi c}{dc^3} + \frac{r^2}{c}\frac{d^5\psi c}{dc^5} + \dots\right) = 0, \text{ whatever } r \text{ be};$$

$$\therefore\ \frac{d}{dc}\left(\frac{1}{c}\frac{d^3\psi c}{dc^3}\right) = 0, \quad \frac{d}{dc}\left(\frac{1}{c}\frac{d^5\psi c}{dc^5}\right) = 0, \dots$$

But $\dfrac{d\psi c}{dc} = c\int\phi(c)\, dc$, $\dfrac{d^2\psi c}{dc^2} = \int\phi(c)\, dc + c\phi(c)$,

$$\frac{d^3\psi c}{dc^3} = 2\phi c + c\frac{d\phi c}{dc};$$

therefore by the first of the above equations of condition

$$\frac{2}{c}\phi c + \frac{d\phi c}{dc} = \text{const.} = 3A,$$

and multiplying by c^2 and integrating

$$c^2 \phi(c) = Ac^3 + B,$$

A and B being independent of c,

$$\phi(c) = Ac + \frac{B}{c^2}.$$

This is the most general solution of the first of the equations of condition for $\psi(c)$, and it satisfies all the rest. Hence the only laws of attraction which have the property in question are those of the direct distance, the inverse square, and a law compounded of these.

PROP. *To find for what laws the shell attracts an internal point equally in every direction.*

10. When this is the case

$$\frac{d}{dc}\left\{\frac{\psi(r+c) - \psi(r-c)}{c}\right\} = 0,$$

$$\frac{d\psi r}{dr} + \frac{d^3\psi r}{dr^3}\frac{c^2}{1.2.3} + \ldots = -A,$$

whatever c is, A being a constant independent of c;

$$\therefore \frac{d\psi r}{dr} = -A, \quad \frac{d^3\psi r}{dr^3} = 0, \ldots$$

These conditions are all satisfied if the first is: this gives

$$r\int\phi(r)\,dr = -A, \quad \phi(r) = \frac{A}{r^2},$$

and therefore the inverse square is the only law which possesses this property.

11. As the form of the Earth and of the other bodies of the Solar System differs from the spherical, and more resembles the spheroidal, it is desirable to find the attraction of a spheroid upon an external and an internal point.

PROP. *To find the attraction of a homogeneous oblate spheroid upon a particle within its mass; the law of attraction being that of the inverse square of the distance.*

12. Let a, c be the semi-axes; the minor axis $2c$ coinciding with the axis of z: then the equation to the spheroid from the centre is

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1.$$

Let fgh be the co-ordinates to the attracted particle, which we shall take as the origin of polar co-ordinates,

$r=$ radius vector of any particle of the attracting mass,

$\theta=$ angle which r makes with a line parallel to z,

$\phi=$ angle which the plane in which θ is measured makes with the plane xz;

$\therefore\ x=f+r\sin\theta\cos\phi,\ y=g+r\sin\theta\sin\phi,\ z=h+r\cos\theta,$

and the equation to the spheroid becomes

$$\frac{(f+r\sin\theta\cos\phi)^2+(g+r\sin\theta\sin\phi)^2}{a^2}+\frac{(h+r\cos\theta)^2}{c^2}=1,$$

$$\text{or } r^2\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}\right)+2r\left(\frac{f\sin\theta\cos\phi+g\sin\theta\sin\phi}{a^2}+\frac{h\cos\theta}{c^2}\right)$$

$$=1-\frac{f^2+g^2}{a^2}-\frac{h^2}{c^2};$$

$$\text{put}\quad \frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}=K,$$

$$\frac{f\sin\theta\cos\phi+g\sin\theta\sin\phi}{a^2}+\frac{h\cos\theta}{c^2}=F,$$

$$\text{and}\quad F^2+K\left(1-\frac{f^2+g^2}{a^2}-\frac{h^2}{c^2}\right)=H,$$

$$\text{then}\quad K^2r^2+2KFr+F^2=H,$$

and the values of r are

$$r'=\frac{-F+\surd(H)}{K}\text{ and }r''=\frac{-F-\surd(H)}{K}.$$

The volume of the attracting element $= r^2 \sin\theta\, dr\, d\theta\, d\phi$ as in Art. 2: let ρ be the density of the spheroid. Then the attraction of this element on the attracted particle is

$$\rho \sin\theta\, dr\, d\theta\, d\phi:$$

and the resolved parts of this parallel to the axes of xyz are

$$\rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi, \quad \rho \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi,$$

$$\rho \sin\theta \cos\theta\, dr\, d\theta\, d\phi.$$

Let A, B, C be the attractions of the whole spheroid in the directions of the axes, estimated positive towards the centre of the spheroid. Then these equal the integrals of the attractions of the element; the limits of r being $-r'$ and r'', of θ being 0 and π, of ϕ being 0 and π. Hence

$$A = -\int_{-r'}^{r''}\int_0^\pi\int_0^\pi \rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi,$$

$$B = -\int_{-r'}^{r''}\int_0^\pi\int_0^\pi \rho \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi,$$

$$C = -\int_{-r'}^{r''}\int_0^\pi\int_0^\pi \rho \sin\theta \cos\theta\, dr\, d\theta\, d\phi.$$

Then $$A = -\rho\int_0^\pi\int_0^\pi (r'' + r') \sin^2\theta \cos\phi d\theta\, d\phi$$

$$= 2\rho\int_0^\pi\int_0^\pi \frac{F}{K} \sin^2\theta \cos\phi d\theta\, d\phi.$$

Now it is easily seen that if $R(\sin\alpha, \cos^2\alpha)$ be a rational function of $\sin\alpha$ and $\cos^2\alpha$, then

$$\int_0^\pi R(\sin\alpha, \cos^2\alpha) \cos\alpha d\alpha = 0.$$

Therefore by substituting for F and K we have

$$A = 2f\rho c^2 \int_0^\pi\int_0^\pi \frac{\sin^3\theta \cos^2\phi\, d\theta\, d\phi}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

$$= \pi f \rho c^2 \int_0^\pi \frac{\sin^3 \theta d\theta}{c^2 \sin^2 \theta + a^2 \cos^2 \theta} = \pi f \rho c^2 \int_0^\pi \frac{(1 - \cos^2 \theta) \sin \theta d\theta}{c^2 + (a^2 - c^2) \cos^2 \theta} *$$

$$= \pi f \rho \frac{c^2}{a^2 - c^2} \int_0^\pi \left\{ \frac{a^2 \sin \theta}{c^2 + (a^2 - c^2) \cos^2 \theta} - \sin \theta \right\} d\theta$$

$$= \pi f \rho \frac{c^2}{a^2 - c^2} \left\{ - \frac{a^2}{c\sqrt{a^2 - c^2}} \tan^{-1} \left(\frac{\sqrt{a^2 - c^2}}{c} \cos \theta \right) + \cos \theta + \text{const.} \right\}$$

$$= 2\pi f \rho \frac{c^2}{a^2 - c^2} \left\{ \frac{a^2}{c\sqrt{a^2 - c^2}} \tan^{-1} \frac{\sqrt{a^2 - c^2}}{c} - 1 \right\}, \quad \frac{c^2}{a^2} = 1 - e^2,$$

$$= 2\pi f \rho \left\{ \frac{\sqrt{1 - e^2}}{e^3} \tan^{-1} \frac{e}{\sqrt{1 - e^2}} - \frac{1 - e^2}{e^2} \right\}$$

$$= 2\pi f \rho \left\{ \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e - \frac{1 - e^2}{e^2} \right\}.$$

In the same manner we should find that

$$B = 2\pi g \rho \left\{ \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e - \frac{1 - e^2}{e^2} \right\}.$$

Also $\quad C = 2\rho \int_0^\pi \int_0^\pi \frac{F}{K} \sin \theta \cos \theta \, d\theta \, d\phi$

$$= 2\rho h a^2 \int_0^\pi \int_0^\pi \frac{\sin \theta \cos^2 \theta \, d\theta \, d\phi}{c^2 \sin^2 \theta + a^2 \cos^2 \theta}$$

$$= 2\pi \rho h \frac{a^2}{a^2 - c^2} \int_0^\pi \left\{ \sin \theta - \frac{c^2 \sin \theta}{c^2 + (a^2 - c^2) \cos^2 \theta} \right\} d\theta$$

$$= 4\pi \rho h \frac{a^2}{a^2 - c^2} \left\{ 1 - \frac{c}{\sqrt{a^2 - c^2}} \tan^{-1} \frac{\sqrt{a^2 - c^2}}{c} \right\}$$

$$= 4\pi \rho h \left\{ \frac{1}{e^2} - \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e \right\}.$$

* If the spheroid be prolate, c is $> a$, and the denominator of this must be written $c^2 - (c^2 - a^2) \cos^2 \theta$, and the integral would involve logarithms instead of circular arcs.

13. COR. 1. We gather from these expressions, that the attraction is independent of the magnitude of the spheroid, and depends solely upon its ellipticity. Hence the attraction of any other spheroid similar to the given one, and comprising the attracted particle in its mass, is the same as that of the given spheroid. Hence a spheroidal shell, the surfaces of which are similar and concentric, attracts a point within it equally in all directions.

The same geometrical proof of this property may be given as in Art. 5.

14. COR. 2. If we put the ellipticity of the spheroid $=\epsilon$, and suppose ϵ so small that we may neglect its square, we have

$$e^2 = 1 - \frac{c^2}{a^2} = 1 - (1-\epsilon)^2 = 2\epsilon;$$

$$\therefore A = \frac{4}{3}\pi\rho\left(1 - \frac{2}{5}\epsilon\right)f, \quad B = \frac{4}{3}\pi\rho\left(1 - \frac{2}{5}\epsilon\right)g,$$

$$C = \frac{4}{3}\pi\rho\left(1 + \frac{4}{5}\epsilon\right)h.$$

If we had taken an ellipsoid instead of a spheroid, the expressions would not have been capable of integration.

15. If we had attempted to find the attraction on an external particle according to the process of the last Proposition, we should have fallen upon expressions which no known methods have yet integrated: and therefore we are unable by any direct means to obtain the attraction of a spheroid on an external particle. Mr Ivory has, however, devised an indirect method of obtaining it, which we shall now proceed to develop. He has discovered a theorem by which the attraction of an ellipsoid upon an external particle is shown to be proportional to that of another ellipsoid, dependent on the first for form and dimensions, upon a particle internal to it, and therefore (in the case of a spheroid, or ellipsoid of revolution) determinable by the last Proposition.

PROP. *To enunciate and prove Ivory's Theorem.*

16. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$,

be the equations to the surfaces of two ellipsoids having the same centre and foci: then

$$a^2-b^2=\alpha^2-\beta^2,\quad a^2-c^2=\alpha^2-\gamma^2\ldots\ldots\ldots\ldots(1).$$

Let fgh, $f'g'h'$ be the co-ordinates to two particles so situated on the surfaces of these ellipsoids that

$$\frac{f}{f'}=\frac{a}{\alpha},\quad \frac{g}{g'}=\frac{b}{\beta},\quad \frac{h}{h'}=\frac{c}{\gamma}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also since (fgh) $(f'g'h')$ are points in the surfaces of the first and second ellipsoids respectively, we have

$$\frac{f^2}{a^2}+\frac{g^2}{b^2}+\frac{h^2}{c^2}=1,\quad \frac{f'^2}{\alpha^2}+\frac{g'^2}{\beta^2}+\frac{h'^2}{\gamma^2}=1\ldots\ldots\ldots\ldots(3).$$

Then the attraction of the first ellipsoid parallel to the axis of x on the particle at the point $(f'g'h')$ on the surface of the second, is to the attraction of the second ellipsoid on the particle at the point (fgh) on the surface of the first in the same direction, as $ab : \alpha\beta$, the law of attraction being any function of the distance: and similarly with respect to the axes of y and z. This is Ivory's Theorem.

We shall, for convenience, represent the law of attraction by the function $r\phi(r^2)$, r being the distance.

The attraction of the first ellipsoid on the particle $(f'g'h')$ parallel to the axis of z

$$=\rho\iiint(h'-z)\,\phi\{(f'-x)^2+(g'-y)^2+(h'-z)^2\}\,dx\,dy\,dz,$$

the limits of z are $-c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$, and $c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$,

the limits of y are $-b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$, and $b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$,

and the limits of x are $-a$ and a

$$=\rho\iint[\psi\{(f'-x)^2+(g'-y)^2+(h'+z)^2\}$$
$$-\psi\{(f'-x)^2+(g'-y)^2+(h'-z)^2\}]\,dx\,dy$$

between the specified limits:

$$\psi(r)=\frac{1}{2}\int\phi(r)\,dr:$$

it must be remembered that in this expression

$$z = c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)},$$

but we do not substitute this value merely that the function may be preserved under as simple a form as possible. Now put $x = ar$, $y = bs$, $z = ct$, then the attraction

$$= \rho ab \iint [\psi \{(f' - ar)^2 + (g' - bs)^2 + (h' + ct)^2\}$$
$$- \psi \{(f' - ar)^2 + (g' - bs)^2 + (h' - ct)^2\}]\, dr\, ds,$$

the limits of s being $-\sqrt{(1 - r^2)}$ and $\sqrt{(1 - r^2)}$, and those of r being -1 and 1: also $t = \sqrt{(1 - r^2 - s^2)}$.

Now $(f' - ar)^2 + (g' - bs)^2 + (h' \pm ct)^2$

$$= f'^2 + g'^2 + h'^2 - 2(f'ar + g'bs \pm h'ct) + a^2r^2 + b^2s^2 + c^2t^2,$$

substituting for h'^2 by (3) and for t^2,

$$= f'^2\left(1 - \frac{\gamma^2}{\alpha^2}\right) + g'^2\left(1 - \frac{\gamma^2}{\beta^2}\right) + \gamma^2 - 2(f'ar + g'bs \pm h'ct)$$
$$+ (a^2 - c^2)r^2 + (b^2 - c^2)s^2 + c^2,$$

eliminating $f'g'h'$ by (2) and making use of (1),

$$= \frac{f^2}{a^2}(a^2 - c^2) + \frac{g^2}{b^2}(b^2 - c^2) + c^2 - 2(f\alpha r + g\beta s \pm h\gamma t)$$
$$+ (\alpha^2 - \gamma^2)r^2 + (\beta^2 - \gamma^2)s^2 + \gamma^2$$

$$= f^2 + g^2 + h^2 - 2(f\alpha r + g\beta s \pm h\gamma t) + \alpha^2r^2 + \beta^2s^2 + \gamma^2t^2, \text{ by (3)},$$
$$= (f - \alpha r)^2 + (g - \beta s)^2 + (h \pm \gamma t).$$

Hence the attraction of the First Ellipsoid on $(f'g'h')$ parallel to z

$$= \rho ab \iint [\psi \{(f - \alpha r)^2 + (g - \beta s)^2 + (h + \gamma t)^2\}$$
$$- \psi \{(f - \alpha r)^2 + (g - \beta s)^2 + (h - \gamma t)^2\}]\, dr\, ds$$

$= \dfrac{ab}{\alpha\beta} \times$ attraction of Second Ellipsoid on (fgh) in the same direction.

The same may be proved for the attractions parallel to the other axes: and consequently the Theorem, as enunciated, is true.

We may observe that one of these ellipsoids must necessarily be wholly within the other. For if not, the points in which they cut each other lie in the line of which the equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{and} \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1.$$

Suppose a less than α; the points of intersection must satisfy the equation

$$x^2\left(\frac{1}{a^2}-\frac{1}{\alpha^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{\beta^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{\gamma^2}\right)=0;$$

and this by (1) becomes

$$\left(\frac{x}{a\alpha}\right)^2+\left(\frac{y}{b\beta}\right)^2+\left(\frac{z}{c\gamma}\right)^2=0,$$

an equation which can be satisfied only by $x=0$, $y=0$, $z=0$. But these do not satisfy the equations above; and therefore the surfaces do not intersect in any point.

To find the attraction of any ellipsoid of which the semi-axes are a, b, c upon an external point $(f'g'h')$ by the help of this Theorem, we must first calculate the attraction of an ellipsoid of which the semi-axes are $\alpha\beta\gamma$, determined by equations (1) and the second of (3), on an internal point (fgh), f, g and h being given by equations (2). And then the attractions required will be those multiplied by

$$\frac{bc}{\beta\gamma},\ \frac{ac}{\alpha\gamma},\ \frac{ab}{\alpha\beta},\ \text{respectively.}$$

CHAPTER II.

LAPLACE'S COEFFICIENTS AND FUNCTIONS.

17. In the present Chapter we shall develop the properties of those remarkable quantities which have received the name of their great discoverer, under the designation of Laplace's Coefficients and Functions. To do this it will be necessary to anticipate the subject of the following Chapter, and to bring in here a Proposition which should properly stand at the head of that division of this treatise.

Prop. *To obtain formulæ for the calculation of the attraction of a heterogeneous mass upon any particle.*

18. Let ρ be the density of the body at the point (xyz); fgh the co-ordinates of the attracted particle; and, as before, suppose that A, B, C are the attractions parallel to the axes x, y, z, estimated positive towards the origin of co-ordinates. Then

$$A=\iiint\frac{\rho\,(f-x)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$B=\iiint\frac{\rho\,(g-y)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$C=\iiint\frac{\rho\,(h-z)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

the limits being determined by the equation to the surface of the body.

Let $$V=\iiint\frac{\rho\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{1}{2}}};$$

$$\therefore A=-\frac{dV}{df},\quad B=-\frac{dV}{dg},\quad C=-\frac{dV}{dh}.$$

19. It follows, then, that the calculation of the attractions A, B, C depends upon that of V. This function cannot be

calculated except when expanded into a series. It is a function of great importance in Physics: and, for the sake of a name, has been denominated the *Potential* of the attracting mass, as upon its value the amount of the attractive force of the body depends.

20. If s be the length of any line terminating in the point fgh, the attraction at that point resolved along the tangent to s

$$= -\frac{df}{ds}\frac{dV}{df} - \frac{dg}{ds}\frac{dV}{dg} - \frac{dh}{ds}\frac{dV}{dh} = -\frac{dV}{ds}.$$

21. As a particle moves along under the action of the force, the aggregate of the force at each point of its course multiplied by the space through which it moves

$$= \int \frac{dV}{ds}\, ds = V.$$

This furnishes a further reason for the name given to V.

PROP. *To prove that* $\frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = 0$, *or* $-4\pi\rho'$, *according as the attracted particle is not or is part of the mass itself; ρ' being the density of the attracted particle in the latter case.*

22. By differentiating V, we have

$$\frac{dV}{df} = \iiint \frac{-\rho\,(f-x)\,dx\,dy\,dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{3}{2}}},$$

$$\frac{d^2V}{df^2} = \iiint \frac{\rho\,\{2\,(f-x)^2 - (g-y)^2 - (h-z)^2\}\,dx\,dy\,dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{5}{2}}}.$$

In the same manner we shall have

$$\frac{d^2V}{dg^2} = \iiint \frac{\rho\,\{2\,(g-y)^2 - (f-x)^2 - (h-z)^2\}\,dx\,dy\,dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{5}{2}}},$$

$$\frac{d^2V}{dh^2} = \iiint \frac{\rho\,\{2\,(h-z)^2 - (f-x)^2 - (g-y)^2\}\,dx\,dy\,dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{5}{2}}};$$

$$\therefore \frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = \iiint \frac{0 \times dx\,dy\,dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{3}{2}}}.$$

23. When the attracted particle is not a portion of the attracting mass itself, then xyz will never equal fgh respectively, and consequently the expression under the signs of integration vanishes for every particle of the mass:

$$\therefore \frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = 0.$$

This equation was first given by Laplace: and Poisson was the first who showed that it is not true when the attracted particle is part of the attracting mass. In that case the denominator of the fraction under the signs of integration vanishes, and the fraction becomes $\frac{0}{0}$, when $x=f$, $y=g$, $z=h$.

To determine the value in that case, suppose a sphere described in the body, so that it shall include the attracted particle; and let $V = U + U'$, U referring to the sphere, and U' to the excess of the body over the sphere. Then, by what is already proved,

$$\frac{d^2U'}{df^2} + \frac{d^2U'}{dg^2} + \frac{d^2U'}{dh^2} = 0;$$

$$\therefore \frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = \frac{d^2U}{df^2} + \frac{d^2U}{dg^2} + \frac{d^2U}{dh^2}.$$

The centre of the sphere may be chosen as near the attracted particle as we please; and therefore the radius of the sphere may be taken so small that its density may be considered uniform and equal to that at the point (fgh), which we shall call ρ'.

Let $f'g'h'$ be the co-ordinates to the centre of the sphere; then the attractions of the sphere on the attracted point parallel to the axes are, by Art. 3,

$$\frac{4\pi\rho'}{3}(f-f'),\quad \frac{4\pi\rho'}{3}(g-g'),\quad \frac{4\pi\rho'}{3}(h-h'),$$

$$\text{or } -\frac{dU}{df},\quad -\frac{dU}{dg},\quad -\frac{dU}{dh}, \text{ by Art. 20.}$$

$$\therefore \frac{d^2 U}{df^2} + \frac{d^2 U}{dg^2} + \frac{d^2 U}{dh^2} = -4\pi\rho';$$

$$\therefore \frac{d^2 V}{df^2} + \frac{d^2 V}{dg^2} + \frac{d^2 V}{dh^2} = -4\pi\rho',$$

when the attracted particle is within the attracting mass.

24. COR. If V, some known function of the variable co-ordinates fgh, be the potential for the interior of a mass, then the law of its density is given by the formula

$$\text{Density} = -\frac{1}{4\pi}\left(\frac{d^2 V}{df^2} + \frac{d^2 V}{dg^2} + \frac{d^2 V}{dh^2}\right).$$

PROP. *To transform the partial differential equation in V into polar co-ordinates.*

25. Let $r\theta\omega$ be the co-ordinates of (fgh), and $r'\theta'\omega'$ of (xyz), the angles θ and θ' being measured from the axis of z; ω and ω' being the angles which the planes on which θ and θ' are measured make with the plane zx. Then

$$f = r\sin\theta\cos\omega, \quad g = r\sin\theta\sin\omega, \quad h = r\cos\theta,$$

$$x = r'\sin\theta'\cos\omega', \quad g' = r'\sin\theta'\sin\omega', \quad h' = r'\cos\theta'.$$

The first are the same as

$$r^2 = f^2 + g^2 + h^2, \quad \cos\theta = \frac{h}{\sqrt{f^2+g^2+h^2}}, \quad \tan\omega = \frac{g}{f} \ldots\ldots (1);$$

$$\therefore \frac{dV}{df} = \frac{dV}{dr}\frac{dr}{df} + \frac{dV}{d\theta}\frac{d\theta}{df} + \frac{dV}{d\omega}\frac{d\omega}{df},$$

$$\frac{d^2 V}{df^2} = \frac{d}{df}\frac{dV}{dr}\frac{dr}{df} + \frac{d}{df}\frac{dV}{d\theta}\frac{d\theta}{df} + \frac{d}{df}\frac{dV}{d\omega}\frac{d\omega}{df}$$

$$+ \frac{dV}{dr}\frac{d^2 r}{df^2} + \frac{dV}{d\theta}\frac{d^2\theta}{df^2} + \frac{dV}{d\omega}\frac{d^2\omega}{df^2}$$

$$= \frac{d^2 V}{dr^2}\frac{dr^2}{df^2} + \frac{d^2 V}{d\theta^2}\frac{d\theta^2}{df^2} + \frac{d^2 V}{d\omega^2}\frac{d\omega^2}{df^2}$$

$$+ 2\frac{d^2 V}{dr\,d\theta}\frac{dr}{df}\frac{d\theta}{df} + 2\frac{d^2 V}{dr\,d\omega}\frac{dr}{df}\frac{d\omega}{df} + 2\frac{d^2 V}{d\theta\,d\omega}\frac{d\theta}{df}\frac{d\omega}{df}$$

$$+\frac{dV}{dr}\frac{d^2r}{df^2}+\frac{dV}{d\theta}\frac{d^2\theta}{df^2}+\frac{dV}{d\omega}\frac{d^2\omega}{df^2}.$$

The expressions for $\frac{d^2V}{dg^2}$ and $\frac{d^2V}{dh^2}$ are of the same form. These three must be added together and equated to zero. When this is effected the formulæ (1) make

the coefficient of $\frac{d^2V}{dr^2}=\frac{dr^2}{df^2}+\frac{dr^2}{dg^2}+\frac{dr^2}{dh^2}=1,$

the coefficient of $\frac{d^2V}{d\theta^2}=\frac{d\theta^2}{df^2}+\frac{d\theta^2}{dg^2}+\frac{d\theta^2}{dh^2}=\frac{1}{r^2},$

the coefficient of $\frac{d^2V}{d\omega^2}=\frac{d\omega^2}{df^2}+\frac{d\omega^2}{dg^2}+\frac{d\omega^2}{dh^2}=\frac{1}{r^2\sin^2\theta},$

the coefficient of $\frac{d^2V}{dr\,d\theta}=2\frac{dr}{df}\frac{d\theta}{df}+2\frac{dr}{dg}\frac{d\theta}{dg}+2\frac{dr}{dh}\frac{d\theta}{dh}=0,$

the coefficient of $\frac{d^2V}{dr\,d\omega}=2\frac{dr}{df}\frac{d\omega}{df}+2\frac{dr}{dg}\frac{d\omega}{dg}+2\frac{dr}{dh}\frac{d\omega}{dh}=0,$

the coefficient of $\frac{d^2V}{d\theta\,d\omega}=2\frac{d\theta}{df}\frac{d\omega}{df}+2\frac{d\theta}{dg}\frac{d\omega}{dg}+2\frac{d\theta}{dh}\frac{d\omega}{dh}=0,$

the coefficient of $\frac{dV}{dr}=\frac{d^2r}{df^2}+\frac{d^2r}{dg^2}+\frac{d^2r}{dh^2}=\frac{2}{r},$

the coefficient of $\frac{dV}{d\theta}=\frac{d^2\theta}{df^2}+\frac{d^2\theta}{dg^2}+\frac{d^2\theta}{dh^2}=\frac{\cos\theta}{r^2\sin\theta},$

the coefficient of $\frac{dV}{d\omega}=\frac{d^2\omega}{df^2}+\frac{d^2\omega}{dg^2}+\frac{d^2\omega}{dh^2}=0.$

Hence the first side of the equation in V becomes

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}+\frac{1}{r^2}\frac{d^2V}{d\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{dV}{d\theta}+\frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\omega^2},$$

or, multiplying by r^2,

$$r\frac{d^2.rV}{dr^2}+\frac{d^2V}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dV}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2V}{d\omega^2}.$$

Put $\cos\theta = \mu$, then

$$r\frac{d^2 . rV}{dr^2} + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2V}{d\omega^2} = 0, \text{ or } -4\pi\rho' r^2,$$

according as the attracted point is not or is part of the attracting mass.

26. Let R be the reciprocal of the distance of any point of the body from the attracted particle; then

$$R = \{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{-\frac{1}{2}},$$

and it may be shown by precisely the same process as in the previous Articles, that

$$\frac{d^2R}{df^2} + \frac{d^2R}{dg^2} + \frac{d^2R}{dh^2} = 0,$$

and $$r\frac{d^2 . rR}{dr^2} + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2R}{d\omega^2} = 0.$$

PROP. *To explain the method of expanding R in a series.*

27. The expression for R becomes, when the polar coordinates are substituted,

$$[r^2 + r'^2 - 2rr'\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')\}]^{-\frac{1}{2}},$$

and this may be expanded into either of the series

$$\left.\begin{aligned} &P_0\frac{1}{r'} + P_1\frac{r}{r'^2} + \ldots\ldots\ldots + P_i\frac{r^i}{r'^{i+1}} + \ldots \\ \text{or } &P_0\frac{1}{r} + P_1\frac{r'}{r^2} + \ldots\ldots\ldots + P_i\frac{r'^i}{r^{i+1}} + \ldots \end{aligned}\right\} \ldots\ldots (1),$$

where $P_0, P_1, \ldots P_i \ldots$ are all determinate rational and entire functions of

$$\mu, \sqrt{1-\mu^2}\cos\omega, \text{ and } \sqrt{1-\mu^2}\sin\omega;$$

and the same functions of

$$\mu', \sqrt{1-\mu'^2}\cos\omega', \text{ and } \sqrt{1-\mu'^2}\sin\omega'.$$

The general coefficient P_i is of i dimensions in

$$\mu, \sqrt{1-\mu^2}\cos\omega, \text{ and } \sqrt{1-\mu^2}\sin\omega.$$

The greatest value of P_i (disregarding its sign) is unity. For if we put

$$\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega') = \cos\phi = \tfrac{1}{2}\left(z+\frac{1}{z}\right),$$

then P_i = coefficient of c^i in

$$(1+c^2-2c\cos\phi)^{-\frac{1}{2}}, \text{ or } (1-cz)^{-\frac{1}{2}}\left(1-\frac{c}{z}\right)^{-\frac{1}{2}}$$

= coefficient of c^i in

$$\left(1+\frac{1}{2}cz+\frac{1.3}{2.4}c^2z^2+\ldots\right)\left(1+\frac{1}{2}\frac{c}{z}+\frac{1.3}{2.4}\frac{c^2}{z^2}+\ldots\right)$$

$$= A\left(z^i+\frac{1}{z^i}\right)+B\left(z^{i-2}+\frac{1}{z^{i-2}}\right)+\ldots$$

$$= 2A\cos i\phi + 2B\cos(i-2)\phi+\ldots$$

A, B... being all positive and finite. The greatest value of this is, when $\phi = 0$. Hence P_i is greatest when $\phi = 0$.

But then P_i = coefficient of c^i in $(1+c^2-2c)^{-\frac{1}{2}}$ or $(1-c)^{-1}$
= coefficient of c^i in $1+c+c^2+\ldots+c^i+\ldots$
$=1.$

Hence 1 is the greatest value of P_i. It follows that the first or second of series (1) will be convergent according as r is less or greater than r'.

To obtain equations for calculating the coefficients P_0, P_1,... P_i... substitute either of the series (1) in the differential equation in R in the last article, and equate the coefficients of the several powers of r to zero. The general term gives the following equation:

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_i}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2P_i}{d\omega^2}+i(i+1)P_i=0,$$

by integrating which P_i should be determined*. The series for R would then be known.

* For the direct integration of this equation, see two Papers in the *Philosophical Transactions* for 1841 and 1857, by Mr Hargreave and Professor Donkin respectively.

28. The functions $P_0, P_1, \ldots P_i \ldots$ possess some remarkable properties which were discovered by Laplace. They are therefore called, after him, *Laplace's Coefficients*, of the orders 0, 1,... *i*... It will be observed that these quantities are definite and have no arbitrary constants in them. Laplace's Coefficients are therefore certain definite expressions involving only numerical quantities with μ and ω, μ' and ω'. Any other expressions which may satisfy the partial differential equation in P_i, which is called Laplace's Equation, may be designated *Laplace's Functions* to distinguish them from the "Coefficients." The fundamental properties of these Coefficients and Functions we shall now proceed to demonstrate.

PROP. *To prove that if Q_i and $R_{i'}$ be two Laplace's Coefficients or Functions, then* $\int_{-1}^{1}\int_0^{2\pi} Q_i R_{i'}\, d\mu\, d\omega = 0$, *when i and i' are different integers.*

29. By Laplace's Equation in the last Article but one

$$i(i+1)\,Q_i = -\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} - \frac{1}{1-\mu^2}\frac{d^2Q_i}{d\omega^2};$$

$$\therefore \int_{-1}^{1}\int_0^{2\pi} Q_i R_{i'}\, d\mu\, d\omega$$

$$= -\frac{1}{i(i+1)}\int_{-1}^{1}\int_0^{2\pi}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2Q_i}{d\omega^2}\right] R_{i'}\, d\mu\, d\omega.$$

By a double integration by parts

$$\int\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\}R_{i'}d\mu = (1-\mu^2)\frac{dQ_i}{d\mu}R_{i'} - (1-\mu^2)\frac{dR_{i'}}{d\mu}Q_i$$

$$+\int\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\}Q_i\, d\mu;$$

$$\therefore \int_{-1}^{1}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\}R_{i'}\, d\mu = \int_{-1}^{1}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\}Q_i\, d\mu.$$

Again, $\int R_{i'}\frac{d^2Q_i}{d\omega^2}d\omega = R_{i'}\frac{dQ_i}{d\omega} - Q_i\frac{dR_{i'}}{d\omega} + \int Q\frac{d^2R_{i'}}{d\omega^2}d\omega;$

$$\therefore \int_0^{2\pi} R_{i'} \frac{d^2 Q_i}{d\omega^2} d\omega = \int_0^{2\pi} Q_i \frac{d^2 R_{i'}}{d\omega^2} d\omega,$$

since when $\omega = 0$ and 2π, each of the functions Q_i, $R_{i'}$, $\frac{dQ_i}{d\omega}$, $\frac{dR_{i'}}{d\omega}$ has the same values, because they are functions of

$$\mu, \sqrt{1-\mu^2} \cos\omega \text{ and } \sqrt{1-\mu^2} \sin\omega.$$

Hence, $\int_{-1}^{1} \int_0^{2\pi} Q_i R_{i'} \, d\mu \, d\omega$

$$= -\frac{1}{i(i+1)} \int_{-1}^{1} \int_0^{2\pi} \left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2 R_{i'}}{d\omega^2}\right] Q_i \, d\mu \, d\omega$$

$$= \frac{i'(i'+1)}{i(i+1)} \int_{-1}^{1} \int_0^{2\pi} Q_i R_{i'} \, d\mu \, d\omega,$$

by Laplace's Equation.

Hence, $\int_{-1}^{1} \int_0^{2\pi} Q_i R_{i'} \, d\mu \, d\omega = 0$, when i and i' are unequal. When they are the same the equation becomes an identical one, and therefore gives no result.

30. This property is true also when $i = 0$, as may easily be shown by going through the process of the last Proposition, Q_i being Q_0 or a constant.

PROP. *To prove that a function of* μ, $\sqrt{1-\mu^2}\cos\omega$, *and* $\sqrt{1-\mu^2}\sin\omega$, *as* $F(\mu, \omega)$, *can be expanded in a series of Laplace's Functions; provided that* $F(\mu, \omega)$ *do not become infinite between the limits* -1 *and* 1 *of* μ, *and* 0 *and* 2π *of* ω.

31. This very important Proposition will occupy the present and four following Articles.

Let $\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega') = p$: then by Art. 27,

$$(1 + c^2 - 2cp)^{-\frac{1}{2}} = 1 + P_1 c + P_2 c^2 + \ldots\ldots P_i c^i + \ldots\ldots$$

c being any quantity not greater than unity.

Differentiate with respect to c,

$$\frac{p-c}{(1+c^2-2cp)^{\frac{3}{2}}}=P_1+2P_2c+\ldots\ldots+iP_ic^{i-1}+\ldots\ldots$$

Multiply this by $2c$ and add to it the former equation;

$$\therefore \frac{1-c^2}{(1+c^2-2cp)^{\frac{3}{2}}}=1+3P_1c+5P_2c^2+\ldots\ldots+(2i+1)P_ic_i+\ldots$$

Now c being quite arbitrary we may put it $=1$. Then the fraction on the left-hand side of this equation vanishes, except when $p=1$; in which case the fraction on the left hand becomes apparently indeterminate: but it is in reality infinite. For when $p=1$, $\dfrac{1-c^2}{(1+c^2-2cp)^{\frac{3}{2}}}=\dfrac{1+c}{(1-c)^2}=$infinity, when $c=1$.

When $p=1$, then $\mu'=\mu$ and $\omega'=\omega$. For when $p=1$

$$\cos(\omega'-\omega)=\frac{1-\mu\mu'}{\sqrt{(1-\mu^2)(1-\mu'^2)}}=\sqrt{\frac{1-2\mu\mu'+\mu^2\mu'^2}{1-\mu^2-\mu'^2+\mu^2\mu'^2}},$$

and that this may not be greater than unity we must take $\mu^2+\mu'^2$ not greater than $2\mu\mu'$, or $(\mu\sim\mu')^2$ not greater than zero. Hence $\mu'=\mu$, and therefore $\cos(\omega'-\omega)=1$, and $\omega'=\omega$.

Hence, the series $1+3P_1+5P_2+\ldots\ldots+(2i+1)P_i+\ldots\ldots$ vanishes for all values of μ and ω, μ' and ω', except when $\mu=\mu'$ and $\omega=\omega'$, in which case the sum of its terms suddenly changes from zero to infinity.

32. Upon this series depends the important property of Laplace's Functions which we are now demonstrating, and which gives them so great a value in the higher branches of analysis. Our demonstration consists in showing, that

$$\int_{-1}^{1}\int_{0}^{2\pi}\frac{(1-c^2)\,d\mu'\,d\omega'\,F(\mu',\omega')}{(1+c^2-2cp)^{\frac{3}{2}}}=4\pi F(\mu,\omega) \text{ when } c=1,$$

and that consequently

$$4\pi F(\mu,\omega)=\int_{-1}^{1}\int_{0}^{2\pi}(1+3P_1+5P_2+\ldots)\,F(\mu',\omega')\,d\mu'\,d\omega';$$

from which property, as will be seen in the end, our Proposition, as enunciated, immediately flows.

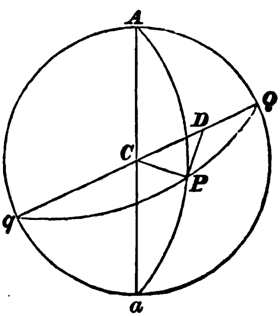
In consequence of the discontinuity above pointed out in the series $1 + 3P_1 + \ldots$, and also because the series becomes infinite in one stage of the variation of its variables, it has been considered by some to be unsatisfactory to deduce any properties from it. But the latter objection is entirely removed by the fact, that we do not use the series in its present form, but after being multiplied by the small infinitesimal quantities $d\mu'$, $d\omega'$ which, as will be seen in the next Art., makes the aggregate of its terms finite, preventing their accumulating to an infinite amount. With regard to the objection of discontinuity, there appears to be no sufficient ground for it. There is no question, that the property deduced (as enunciated in our Proposition) is true, at any rate for rational functions of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$, and is also most important. This objection, however, deserves to be examined with care, which we now propose to do in the course of our demonstration.

33. We shall first prove, that

$$\int_{-1}^{1}\int_{0}^{2\pi}\frac{(1-c^2)\,d\mu'\,d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}}=4\pi.$$

This integration cannot be effected with the co-ordinates as at present chosen. But it may be done by a simple transformation, and a change in the way of taking the elements.

Suppose a sphere of radius unity described about C the origin of co-ordinates. Let θ' and ω' be the angular co-ordinates to a point P, θ' (or $\cos^{-1}\mu'$) measured from a fixed point A along a great circle of the sphere, and ω' the angle which this great circle makes with another and fixed great circle through A. Then $d\theta'.d\omega'\sin\theta'$, or $-d\mu' d\omega'$, is an infinitesimal element of the surface of the sphere at P. This division of the surface into elements supposes it to be cut into lunes from A to a, each being of the angular width $d\omega'$; and these into elements by parallel planes perpendicular



to ACa, at a distance $d\mu'$ from each other. The elements thus formed, though of different shapes, are all equal to each other in area; for the sides are $\sin\theta' d\omega$ and $\operatorname{cosec}\theta' d\mu$, the product of which is $-d\omega d\mu$, which is independent of θ'. Take D a point within the sphere, and let $CD = c$, and suppose CD meets the sphere in Q when produced forwards, and in q when produced backwards. Let μ and ω be the co-ordinates of Q. Then p (see its value, Art. 31) is the cosine of the angle which CP and CQ make with each other: and the distance of P from $D = \sqrt{1 + c^2 - 2cp}$. Let ψ be the angle which the plane CPQ makes with CAQ, that is, the angle AQP. By changing the origin of the angles from A to Q, and dividing the surface of the sphere into new elements in the same way as before beginning from Q as the origin instead of A, the element at P, with these new co-ordinates $\cos^{-1}p$ and ψ, will be $-dpd\psi$, and will $= -d\mu' d\omega'$; for (as stated above) the elements of the dissection beginning from A are all equal in area, though of different shapes. The same also is true of the dissection beginning from Q. Also the dissections from A and Q being precisely similar, the number of elements in the two cases is the same. Hence the element $-d\mu' d\omega'$ at P is equal to $-dp\, d\psi$ at p, though of a different shape.

By reverting to the meaning of integration we see that the integral under consideration $= -(1 - c^2) \times$ limit of sum of all the elements of the surface of the sphere divided respectively by the cubes of their distances from D.

But this, by the change of co-ordinates, also

$$= \int_{-1}^{1} \int_{0}^{2\pi} \frac{(1 - c^2)\, dpd\psi}{(1 + c^2 - 2pp)^{\frac{3}{2}}},$$

which can be at once integrated. It

$$= 2\pi (1 - c^2) \int_{-1}^{1} \frac{dp}{(1 + c^2 - 2cp)^{\frac{3}{2}}}$$

$$= 2\pi \frac{1 - c^2}{c} \frac{1}{\sqrt{1 + c^2 - 2cp}} + \text{const.}$$

$$= 2\pi \frac{1 - c^2}{c} \left(\frac{1}{1 - c} - \frac{1}{1 + c}\right)$$

$$= 2\pi \left(\frac{1 + c}{c} - \frac{1 - c}{c}\right)$$

$$= 4\pi.$$

It is remarkable that this result is altogether independent of c*.

34. By analysing the integral in the last Article and separating it into its elements, we can show by what process c vanishes from the result, and this will assist us in the latter part of the present demonstration. It matters not in which order we effect the integration; we shall therefore integrate with respect to p first, because it becomes necessary to do so in the next Article.

The quantity $1-p$ is the versed-sine of the arc QP, and is measured along the line QCq. Let this line be divided into n parts each equal to dp, so that $n \,.\, dp =$ the diameter $= 2$, n being very large and dp very small. Draw perpendiculars to the diameter through these divisions cutting the circle QPq in a series of points; and call the distances of these points from D, beginning from Q,

$$1-c,\ s',\ s'',\ s'''\ldots\ldots s^{(n-1)},\ 1+c.$$

Suppose P is at the x^{th} division; then by expansion, omitting the squares and higher powers of dp as they vanish in the limit with reference to the first power, we see the truth of the following:

$$\frac{1}{s^{(x)}}-\frac{1}{s^{(x+1)}}=\frac{1}{\sqrt{1+c^2-2cp}}-\frac{1}{\sqrt{1+c^2-2c\,(p+dp)}}$$

$$=\frac{-\,cdp}{(1+c^2-2cp)^{\frac{3}{2}}}\dagger.$$

* This result can be obtained more shortly as follows; but the proof given in the text is necessary for comprehending the remainder of our demonstration. The equivalent of the fraction to be integrated is $(1+3P_1c+5P_2c^2+\ldots)\,d\mu' d\omega'$. The property of Laplace's coefficients proved in Art. 29, shows that every term of this series except the first will vanish in the integration, and the first will give 4π.

† This formula may be proved *geometrically* thus. Draw a diagram according to the following description. On Qq as a diameter describe a semicircle $QPpq$, p being very near P, and C the centre. Take D in Cq, so that $CD=c$. Join DP, Dp, CP. Draw PM, pm perpendicular to Qq, Pn to pm, and pr to DP. Join nr.

Then because the angles at n and r are right angles, a circle can be drawn through P, p, n, r;

By giving x its successive values from 0 to $n-1$, and adding together all the resulting values of the above expression, and afterwards taking the limit, we have the definite integral with respect to p; the limits are from $p=1$ to $p=-1$, or, changing the sign of the integrals, from $p=-1$ to $p=1$. Thus, n being made infinitely great,

$$\int_{-1}^{1}\frac{cdp}{(1+c^2-2cp)^{\frac{3}{2}}}=\left(\frac{1}{1-c}-\frac{1}{s'}\right)+\left(\frac{1}{s'}-\frac{1}{s''}\right)+\ldots$$
$$\ldots+\left(\frac{1}{s^{(n-1)}}-\frac{1}{1+c}\right).$$

Hence,
$$\int_0^{2\pi}\int_{-1}^{1}\frac{(1-c^2)\,d\psi dp}{(1+c^2-2cp)^{\frac{3}{2}}}$$
$$=\int_0^{2\pi}d\psi\left[\frac{1+c}{c}+\frac{1-c^2}{c}\left\{\left(\frac{1}{s'}-\frac{1}{s'}\right)+\ldots\right.\right.$$
$$\left.\left.\ldots+\left(\frac{1}{s^{(n-1)}}-\frac{1}{s^{(n-1)}}\right)\right\}-\frac{1-c}{c}\right]$$
$$=2\pi\left(\frac{1+c}{c}-\frac{1-c}{c}\right)=4\pi,\text{ as before.}$$

Here it will be seen that the terms within the inner brackets mutually destroy each other whatever be the value of c. It may also be observed that were this not the case, that whole part of the expression would vanish for the particular value $c=1$ (which is the only case we shall have to use), whatever the value of the sum of the terms following the multiplier $1-c^2$, so long as that sum is not infinite. This leads us on to the last stage of the demonstration.

$\therefore$ angle Ppr = angle Pnr, or angle CPD = angle Pnr;

also angle nPr = angle PDC;

$\therefore$ rPn and CDP are similar triangles;

$$\therefore\ \frac{Pn}{Pr}=\frac{DP}{DC},\text{ or }\frac{DC\,.\,Mm}{DP}=DP-Dp\text{ ultimately;}$$

$$\therefore\ \frac{DC\,.\,Mm}{DP^3}=\frac{Dp}{DP}\left(\frac{1}{Dp}-\frac{1}{DP}\right)=\frac{1}{Dp}-\frac{1}{DP}\text{ ultimately,}$$

which is the formula in the text. The whole proof of this fundamental principle of Laplace's Functions may therefore be conducted geometrically, as the remainder is already in that form.

35. We have now to show that when $c=1$,

$$\int_{-1}^{1}\int_{0}^{2\pi}(1-c^2)\frac{F(\mu', \omega')\,d\mu'\,d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}}=4\pi F(\mu, \omega).$$

The function $F(\mu', \omega')$ at the point Q is $F(\mu, \omega)$, call it F: let $F', F''\ldots F^{(n)}$ be its values at the points of junction of the successive elements along the great circle QPq. Then by multiplying the successive values of $\frac{1}{s^{(x)}}-\frac{1}{s^{(x+1)}}$ by $F, F', F''\ldots$ dividing them by c, and adding them together, we have

$$\int_{-1}^{1}\int_{0}^{2\pi}(1-c^2)\frac{F(\mu', \omega')\,d\mu'\,d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}},$$

$$\text{or} \int_{0}^{2\pi}\int_{-1}^{1}(1-c^2)\frac{F(\mu', \omega')\,d\psi\,dp}{(1+c^2-2cp)^{\frac{3}{2}}},$$

$$=\int_{0}^{2\pi}d\psi\frac{1-c^2}{c}\left\{\left(\frac{1}{1-c}-\frac{1}{s'}\right)F+\left(\frac{1}{s'}-\frac{1}{s''}\right)F'+\ldots+\left(\frac{1}{s^{(n-1)}}-\frac{1}{1+c}\right)F^{(n-1)}\right\}$$

$$=\int_{0}^{2\pi}d\psi\left[\frac{1+c}{c}F+\frac{1-c^2}{c}\left\{\left(\frac{F'}{s'}-\frac{F}{s'}\right)+\left(\frac{F''}{s''}-\frac{F'}{s''}\right)+\ldots\right\}-\frac{1-c}{c}F^{(n-1)}\right]$$

$$=\int_{0}^{2\pi}d\psi\left[\frac{1+c}{c}F+\frac{1+c}{c}\left\{(F'-F)\frac{1-c}{s'}+(F''-F')\frac{1-c}{s''}+\ldots\right\}-\frac{1-c}{c}F^{(n-1)}\right],$$

n being made infinitely great.

The fractions $\frac{1-c}{s'}$, $\frac{1-c}{s''}$, ... diminish successively in value, being the ratios of QD to the successive values of DP. When $c=1$ each of them vanishes; and none of the factors $F'-F$, $F''-F'$, ... become infinite. Hence when $c=1$ the expression to be integrated becomes $d\psi\,.\,2F(\mu, \omega)$, and the integral of it is $4\pi F(\mu, \omega)$, since μ and ω are altogether independent of ψ. Equating this and the series which represents this same integral,

$$4\pi F(\mu, \omega)=\int_{-1}^{1}\int_{0}^{2\pi}\{1+3P_1+\ldots+(2i+1)P_i+\ldots\}F(\mu', \omega')\,d\mu'\,d\omega';$$

$$\therefore F(\mu, \omega) = \int_{-1}^{1}\int_{0}^{2\pi} \left\{\frac{F(\mu', \omega')}{4\pi} + \frac{3F(\mu', \omega')}{4\pi} P_1 + \dots\dots \right.$$
$$\left. + \frac{(2i+1) F(\mu', \omega')}{4\pi} P_i + \dots\dots \right\} d\mu' d\omega'.$$

The general term of this, viz.

$$\int_{-1}^{1}\int_{0}^{2\pi} \frac{2i+1}{4\pi} F(\mu', \omega') P_i d\mu' d\omega',$$

which we will call F_i, is a function of μ and ω; and evidently satisfies Laplace's Equation in μ and ω, because P_i does so. Hence, this is a Laplace's Function, of the i^{th} order: and the result is, what we were to demonstrate, that any function of μ and ω may be expanded in a series of Laplace's Functions; or,

$$F(\mu, \omega) = F_0 + F_1 + F_2 + \dots\dots + F_i + \dots\dots$$

36. Those who are at all acquainted with the controversy which followed the first discovery of these remarkable functions by Laplace, will understand why we have entered so fully upon the subject. Laplace's demonstration in the *Mécanique Céleste* was by no means conclusive. This Mr Ivory pointed out in the *Philosophical Transactions* for 1812; and in the Volume for 1822 he threw considerable doubt upon the applicability of the theorem to functions that are not rational and entire functions of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$. Poisson wrote much upon the subject. In the first edition of the author's *Mechanical Philosophy* the last method of Poisson was followed, as given in his *Théorie Mathématique de la Chaleur;* in which he effects the integration of the fraction on the left-hand side by the artifice of substituting for it an integrable fraction, but entirely different in its general form, which coincides with it in the particular case for which he requires it in the result, viz. when $c = 1$. In the Second Edition of the *Mechanical Philosophy* we gave a much shorter proof, based upon an idea taken from Professor O'Brien's *Mathematical Tracts.* But this also rather concealed the real difficulty of the case, and passed it over by an artifice. In the demonstration now given, we have gone to the foundation of the calculus, the doctrine of limits, and attempted to clear up all difficulty and ambiguity in the matter.

With regard to the doubt thrown out by Ivory, alluded to above, it seems to be clear that theoretically every function can be expanded in a series of Laplace's Functions: but if it be not a rational function of the co-ordinates, the number of terms in the series will be infinite, and if the terms be not convergent, the expansion, or rather arrangement, will be useless. But this must be determined in each case. A similar uncertainty, requiring examination, always attends the use of infinite series.

PROP. *To prove that a function of μ and ω can be arranged in only one series of Laplace's Functions.*

37. For if possible let both these be true,

$$F(\mu, \omega) = F_0 + F_1 + F_2 + \ldots\ldots + F_i + \ldots\ldots$$

$$F(\mu, \omega) = G_0 + G_1 + G_2 + \ldots\ldots + G_i + \ldots\ldots$$

$$\therefore\ 0 = (F_0 - G_0) + (F_1 - G_1) + \ldots\ldots + (F_i - G_i) + \ldots\ldots$$

and if these letters be accented when μ' and ω' are the variables instead of μ and ω, then

$$0 = (F_0' - G_0') + (F_1' - G_1') + \ldots\ldots + (F_i' - G_i') + \ldots\ldots$$

$$\therefore\ 0 = \int_{-1}^{1}\int_{0}^{2\pi} P_i (F_i' - G_i')\, d\mu' d\omega', \text{ by Art. 29.}$$

But the principle demonstrated in the last Proposition shows that

$$F_i - G_i = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} (1 + 3P_1 + \ldots)\,(F_i' - G_i')\, d\mu' d\omega'$$

$$= \frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} P_i (F_i' - G_i')\, d\mu' d\omega', \text{ by Art. 29,}$$

$$= 0, \text{ by the condition deduced above;}$$

therefore $F_i = G_i$, and the two series are term by term identical, and the Proposition is true.

38. It follows from this, that if by any process we can expand a function in a series of quantities which satisfy Laplace's Equation, that is the only series of the kind into

which it can be expanded: and if by any other process we obtain what is apparently another, the terms of the two series must be the same, term by term, and we may put them equal to each other.

39. Before concluding this Chapter, we shall explain how the numerical coefficients in $P_0 P_1 \ldots P_i \ldots$ are found: and shall give a few examples of the truth of the last Proposition but one (that in Art. 35) by actual integration.

PROP. *To explain how to expand P_i.*

40. By Art. 27 P_i is the coefficient of c^i in the expansion of the function

$$[1 + c^2 - 2c\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')\}]^{-\frac{1}{2}},$$

and is therefore a rational and entire function of μ,

$$\sqrt{1-\mu^2}\cos\omega, \text{ and } \sqrt{1-\mu^2}\sin\omega;$$

and is precisely the same function of μ',

$$\sqrt{1-\mu'^2}\cos\omega', \text{ and } \sqrt{1-\mu'^2}\sin\omega'.$$

The general term of P_i, viz. that involving $\cos n(\omega-\omega')$, can arise solely from the powers n, $n+2$, $n+4$, ... of $\cos(\omega-\omega')$. Hence $(1-\mu^2)^{\frac{n}{2}}$ will occur as a factor of that term: and the other part of its coefficient will be a factor of the form

$$A_0\mu^{i-n} + A_1\mu^{i-n-2} + \ldots + A_s\mu^{i-n-2s} + \ldots = H_n \text{ suppose.}$$

Hence

$$P_i = H_0 + (1-\mu^2)^{\frac{1}{2}}H_1\cos(\omega-\omega') + \ldots + (1-\mu^2)^{\frac{n}{2}}H_n\cos n(\omega-\omega') + \ldots$$

If this be substituted for P_i in Laplace's Equation and the coefficient of $\cos n(\omega-\omega')$ be equated to zero, we obtain a condition from which to calculate the arbitrary constants we have introduced. This condition, after reduction and arrangement, is as follows:

$$0 = (i-n)(i+n+1)\,H_n(1-\mu^2)^n + \frac{d}{d\mu}\left\{(1-\mu^2)^{n+1}\frac{dH_n}{d\mu}\right\}.$$

Substituting in this the series which H_n represents, and equating the coefficient of the general term $(1-\mu^2)^n\mu^{i-n-2s}$ to zero, and reducing, we arrive at the formula

$$A_s = -\frac{(i-n-2s+2)(i-n-2s+1)}{2s(2i-2s+1)} A_{s-1}.$$

By making s successively equal 1, 2, 3 ... we have $A_1 A_2 \ldots$ in terms of A_0. Let these be substituted, and we have the coefficient of $\cos n(\omega - \omega') =$

$$A_0 (1-\mu^2)^{\frac{n}{2}} \left\{ \mu^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} \mu^{i-n-2} + \ldots \right\},$$

call this $A_0 f(\mu)$. The coefficient A_0 is a function of μ', but is independent of μ: and because P_i is the same function of μ' that it is of μ, it follows that $A_0 = \alpha_n f(\mu')$, where α_n is a numerical quantity: and the coefficient of

$$\cos n(\omega - \omega') = \alpha_n f(\mu') f(\mu).$$

To find α_n we must compare the first term of the ascending expansion of $\alpha_n f(\mu') f(\mu)$ in powers of μ with the corresponding term in the coefficient of c^i in the actual expansion of

$$[1 + c^2 - 2c\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega - \omega')\}]^{-\frac{1}{2}}.$$

This leads to the following result:

$$\alpha_n = 2\left\{\frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i}\right\}^2 \frac{i(i-1)\ldots(i-n+1)}{(i+1)(i+2)\ldots(i+n)};$$

this applies when $n = 1, 2, 3 \ldots$, but evidently not when $n = 0$: α_0 is found by equating coefficients to be

$$\left\{\frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i}\right\}^2.$$

We have now the complete value of P_i in a series; it is as follows:

$$P_i = \left\{\frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i}\right\}^2$$

$$\times \left[\left\{ \mu^i - \frac{i(i-1)}{2(2i-1)} \mu^{i-2} + \frac{i(i-1)}{2(2i-1)} \frac{(i-2)(i-3)}{4(2i-3)} \mu^{i-4} - \&c.\ldots \right\} \right.$$

$$\times \left\{ \mu'^i - \frac{i(i-1)}{2(2i-1)} \mu'^{i-2} + \frac{i(i-1)}{2(2i-1)} \frac{(i-2)(i-3)}{4(2i-3)} \mu'^{i-4} - \&c.\ldots \right\}$$

$$+ 2\cos(\omega - \omega') \frac{i}{i+1}$$

$$\times(1-\mu^2)^{\frac{1}{2}}\left\{\mu^{i-1}-\frac{(i-1)(i-2)}{2(2i-1)}\mu^{i-3}+\frac{(i-1)(i-2)}{2(2i-1)}\frac{(i-3)(i-4)}{4(2i-3)}\mu^{i-5}\ldots\right\}$$

$$\times(1-\mu'^2)^{\frac{1}{2}}\left\{\mu'^{i-1}-\frac{(i-1)(i-2)}{2(2i-1)}\mu'^{i-3}+\frac{(i-1)(i-2)}{2(2i-1)}\frac{(i-3)(i-4)}{4(2i-3)}\mu'^{i-5}\ldots\right\}$$

$$+2\cos 2(\omega-\omega')\frac{i(i-1)}{(i+1)(i+2)}$$

$$\times(1-\mu^2)\left\{\mu^{i-2}-\frac{(i-2)(i-3)}{2(2i-1)}\mu^{i-4}+\frac{(i-2)(1-3)}{2(2i-1)}\frac{(i-4)(i-5)}{4(2i-3)}\mu^{i-6}\ldots\right\}$$

$$\times(1-\mu'^2)\left\{\mu'^{i-2}-\frac{(i-2)(i-3)}{2(2i-1)}\mu'^{i-4}+\frac{(i-2)(i-3)}{2(2i-1)}\frac{(i-4)(i-5)}{4(2i-3)}\mu'^{i-6}\ldots\right\}$$

$$+\&c.\ldots\Big].$$

41. The following numerical examples are written down for convenience of reference:

(1) $P_1=\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')$.

(2) $$P_2=\frac{9}{4}\left\{\left(\mu^2-\frac{1}{3}\right)\left(\mu'^2-\frac{1}{3}\right)+\frac{4}{3}(1-\mu^2)^{\frac{1}{2}}\mu(1-\mu'^2)^{\frac{1}{2}}\mu'\cos(\omega-\omega')\right.$$
$$\left.+\frac{1}{3}(1-\mu^2)(1-\mu'^2)\cos 2(\omega-\omega')\right\}.$$

(3) $$P_3=\frac{25}{4}\left\{(\mu^3-\frac{3}{5}\mu)(\mu'^3-\frac{3}{5}\mu')\right.$$
$$+\frac{3}{2}(1-\mu^2)^{\frac{1}{2}}\left(\mu^2-\frac{1}{5}\right)(1-\mu'^2)^{\frac{1}{2}}\left(\mu'^2-\frac{1}{5}\right)\cos(\omega-\omega')$$
$$\left.+\frac{3}{5}(1-\mu^2)\mu(1-\mu'^2)\mu'\cos 2(\omega-\omega')+\frac{1}{10}(1-\mu^2)^{\frac{3}{2}}(1-\mu'^2)^{\frac{3}{2}}\cos 3(\omega-\omega')\right\}$$

&c. = &c.

42. The following are some examples of expanding a function in a series of Laplace's Functions, by an application of the formula

$$F_i = \frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} F(\mu', \omega')\, P_i\, d\mu'\, d\omega',$$

proved in Art. 35.

Ex. 1. Arrange $a + b\mu^2$ in terms of Laplace's Functions.

Here $F(\mu', \omega') = a + b\mu'^2$. First put $i = 0$, $P_0 = 1$;

$$\therefore F_0 = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} (a + b\mu'^2)\, d\mu'\, d\omega' = \frac{1}{2}\int_{-1}^{1} (a + b\mu'^2)\, d\mu'$$

$$= \frac{1}{2}\left(a\mu' + \frac{1}{3} b\mu'^3 + \text{const.}\right) = a + \frac{1}{3} b.$$

Again, put $i = 1$, P_1 is found in the last Article.

$$\therefore F_1 = \frac{3}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} (a + b\mu'^2)\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')\}\, d\mu'\, d\omega'$$

$$= \frac{3}{4\pi}\int_{-1}^{1} (a + b\mu'^2)\{\mu\mu' . \omega' - \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\sin(\omega-\omega')\}\, d\mu',$$

between the proper limits, $\omega' = 0$ and $\omega' = 2\pi$,

$$= \frac{3}{2}\int_{-1}^{1} (a + b\mu'^2)\, \mu\mu'\, d\mu' = \frac{3}{2}\mu\left(\frac{1}{2} a\mu'^2 + \frac{1}{4} b\mu'^4\right),$$

between the limits $\mu' = -1$ and $\mu' = 1$, $= 0$.

Next, put $i = 2$, and substitute for P_2 from the last Article.

$$\therefore F_2 = \frac{5}{4\pi}\int_{-1}^{1}\int_{0}^{2} (a + b\mu'^2)\left\{\frac{9}{4}\left(\mu^2 - \frac{1}{3}\right)\left(\mu'^2 - \frac{1}{3}\right) + M\cos(\omega-\omega')\right.$$

$$\left. + N\cos 2(\omega-\omega')\right\} d\mu'\, d\omega$$

$$= \frac{5}{2}\int_{-1}^{1} (a + b\mu'^2)\, \frac{9}{4}\left(\mu^2 - \frac{1}{3}\right)\left(\mu'^2 - \frac{1}{3}\right) d\mu'$$

$$=\frac{45}{8}\left(\mu^2-\frac{1}{3}\right)\int_{-1}^{1}\left\{-\frac{1}{3}a+(a-\frac{1}{3}b)\,\mu'^2+b\mu'^4\right\}d\mu'$$

$$=\frac{45}{8}\left(\mu^2-\frac{1}{3}\right)\left\{-\frac{2}{3}a+\frac{2}{3}(a-\frac{1}{3}b)+\frac{2}{5}b\right\}=b\left(\mu^2-\frac{1}{3}\right).$$

Hence the function $a+b\mu^2$ stands as follows, when arranged in terms of Laplace's Functions,

$$(a+\frac{1}{3}b)+b\left(\mu^2-\frac{1}{3}\right),$$

and consists of two Functions, of the order 0 and 2 respectively. The above is a long process to arrive at this result. It might have been so arranged at a glance. But the calculation has been given as an example of the use of the formula, which in most cases is the only means of obtaining the desired result.

Ex. 2. Arrange $49+30\mu+3\mu^2+\sqrt{1-\mu^2}\,(40+72\mu)\cos(\omega-\alpha)$ $+\,24\,(1-\mu^2)\cos 2(\omega-\alpha)$ in terms of Laplace's Functions.

The result is $50+\{30\mu+40\sqrt{1-\mu^2}\cos(\omega-\alpha)\}$

$+\,\{3\mu^2-1+72\mu\sqrt{1-\mu^2}\cos(\omega-\alpha)+24\,(1-\mu^2)\cos 2\,(\omega-\alpha)\}$,

consisting of three functions of the orders, 0, 1, 2.

Ex. 3. Let the function be

$$1+\sqrt{2-2\mu^2}\cos(\omega+\alpha)+\frac{1}{2}(1-\mu^2)\cos 2\,(\omega+\alpha).$$

The first term is a Laplace's Function of the order 0, and the second and third terms taken together are one of the second order.

Ex. 4. Let $1-(1-\mu^2)\cos^2\omega$ be the function. The arrangement is

$$\frac{2}{3}+\frac{1}{2}\left\{\left(\mu^2-\frac{1}{3}\right)-(1-\mu^2)\cos 2\omega\right\},$$

or, which is the same,

$$\frac{2}{3}+\left\{\frac{1}{3}-(1-\mu^2)\cos^2\omega\right\}.$$

43. In the preceding Articles P_i has been expanded in terms of μ and ω. But it admits of a simpler expansion in terms of p. Thus, by Art. 27,

$$(1+c^2-2cp)^{-\frac{1}{2}}=1+P_1c+\ldots+P_ic^i+\ldots$$

So if M_i is the same function of μ that P_i is of p, then

$$(1+c^2-2c\mu)^{-\frac{1}{2}}=1+M_1c+\ldots+M_ic^i+\ldots$$

But p becomes μ, and the first of these series becomes the second, and P_i becomes M_i, when μ' is put $=1$ in the expression for p, viz. $\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')$. Put, then, $\mu'=1$ in the value of P_i in Art. 40 and we have the expansion of M_i in terms of μ; and then putting p for μ we have

$$P_i=\left\{\frac{1.3\ldots(2i-1)}{1.2\ldots i}\right\}^2\left\{1-\frac{i(i-1)}{2(2i-1)}+\frac{i(i-1)}{2(2i-1)}\frac{(i-2)(i-3)}{4(2i-2)}+\ldots\right\}$$

$$\times\left\{p^i-\frac{i(i-1)}{2(2i-1)}p^{i-2}+\ldots\right\}.$$

CHAPTER III.

ATTRACTION OF BODIES NEARLY SPHERICAL.

44. As the Earth and other bodies of the Solar System are nearly spherical, and yet may not be precisely of the spheroidal form, it is found necessary in questions of Physical Astronomy to calculate the attraction of bodies nearly spherical. In these calculations is seen the value of the Functions we have been considering in the last Chapter.

If $r'\theta'\omega'$ be the co-ordinates to any element of the attracting mass, ρ' be its density, and $\cos\theta' = \mu'$, then the mass of this element

$$= \rho' dr' r' d\theta' r' \sin\theta' d\omega' = -\rho' r'^2 dr' d\mu' d\omega',$$

and the reciprocal of the distance being R, by Art. 18 and 27, the potential V

$$= \int_0^r \int_{-1}^1 \int_0^{2\pi} \rho' \left(P_0 \frac{r'^2}{r} + P_1 \frac{r'^3}{r^2} + \dots + P_i \frac{r'^{i+2}}{r^{i+1}} + \dots \right) dr' d\mu' d\omega';$$

$$\text{or} \int_0^r \int_{-1}^1 \int_0^{2\pi} \rho' \left(P_0 r' + P_1 r + P_2 \frac{r^2}{r'} + \dots + P_i \frac{r^i}{r'^{i-1}} + \dots \right) dr' d\mu' d\omega',$$

according as r, the distance of the attracted point from the origin, is greater or less than r'. We shall proceed soon to use these formulæ; but we must first find the value of the potential V for a perfect sphere.

PROP. *To calculate the value of V for a homogeneous sphere.*

45. Let the centre of the sphere be the origin of the polar co-ordinates $(r'\mu'\omega')$ to any element of its mass, and the line through the attracted point be that from which the angles are

measured, and ρ the density. Then $-\rho r'^2 dr'\, d\mu'\, d\omega'$ is the mass of the element: its distance from the attracted point

$$=\sqrt{r^2+r'^2-2rr'\cos\omega}.$$

Hence, a being the radius of the sphere,

$$V=\int_0^a\int_{-1}^1\int_0^{2\pi}\frac{\rho r'^2\, dr'\, d\mu'\, d\omega'}{\sqrt{r^2+r'^2-2rr'\mu'}}$$

$$=2\pi\rho\int_0^a\int_{-1}^1\frac{r'^2\, dr'\, d\mu'}{\sqrt{r^2+r'^2-2rr'\mu'}}=2\pi\rho\int_0^a-\frac{r'}{r}\left\{\sqrt{r^2+r'^2-2rr'\mu'}\right\}dr',$$

$$\text{from } \mu'=-1 \text{ to } \mu'=1,\ =2\pi\rho\int_0^a\frac{r'}{r}\left\{(r+r')\mp(r-r')\right\}dr',$$

$-$ when the attracted point is without, and $+$ when it is within the shell,

$$=\frac{4\pi\rho}{r}\int_0^a r'^2\, dr'=\frac{4\pi\rho a^3}{3r},$$

when the point is without the sphere.

When the point is within the sphere, the part of V for the shells which enclose the point

$$=2\pi\rho\int_r^a 2r'\, dr'=2\pi\rho\,(a^2-r^2):$$

and the part of V for the other shells of the sphere

$$=\frac{4\pi\rho}{r}\int_0^r r'^2\, dr'=\frac{4}{3}\pi\rho r^2.$$

Hence $V=\dfrac{4\pi\rho a^3}{3r}$ for an *external* particle,

$$V=2\pi\rho a^2-\frac{2}{3}\pi\rho r^2 \text{ for an internal particle.}$$

PROP. *To find the attraction of a homogeneous body, differing little from a sphere in form, on a particle without it.*

46. Since the attracted particle is without the attracting mass, we must expand V in a descending series of powers of r, and shall therefore use the first of the expressions for V in Art. 44. Let the mean radius of the body $= \mathrm{a}$; and let $\mathrm{a}(1+y')$ be the variable radius, y' being a function of μ' and ω', and its square being neglected.

Then, for the excess of the attracting mass over the sphere of which the radius $= \mathrm{a}$, effecting the integration with respect to r' from $r' = \mathrm{a}$ to $r' = \mathrm{a}(1+y')$, the value of V

$$=\rho\int_{-1}^{1}\int_{0}^{2\pi}\left\{\frac{\mathrm{a}^3}{r}P_0+\frac{\mathrm{a}^4}{r^2}P_1+\ldots+\frac{\mathrm{a}^{i+3}}{r^{i+1}}P_i+\ldots\right\}y'\,d\mu'\,d\omega'.$$

But if y, the same function of μ and ω that y' is of μ' and ω', be expanded in a series of Laplace's Functions, viz.

$$Y_0+Y_1+\ldots+Y_i+\ldots,$$

then the theorems of Art. 29 and 35 show that

$$\frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}y'P_i\,d\mu'd\omega'=\frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}Y_i'P_i\,d\mu'd\omega'=Y_i.$$

Hence the value of V for the excess over the sphere becomes

$$=\frac{4\pi\rho\mathrm{a}^3}{r}\left\{Y_0+\frac{\mathrm{a}}{3r}Y_1+\ldots+\frac{\mathrm{a}^i}{(2i+1)r^i}Y_i+\ldots\right\};$$

and the part of V for the sphere, rad. $= \mathrm{a}$, is

$$V=\frac{4\pi\rho\mathrm{a}^3}{3r}.$$

Hence for the whole mass

$$V=\frac{4\pi\rho\mathrm{a}^3}{3r}+\frac{4\pi\rho\mathrm{a}^3}{r}\left\{Y_0+\frac{\mathrm{a}}{3r}Y_1+\ldots+\frac{\mathrm{a}^i}{(2i+1)r^i}Y_i+\ldots\right\}.$$

47. This is the first example in which we see the great value of the properties of Laplace's Functions; they here

give us at once the integrals involved in our expression for V, in terms of the equation to the surface of the attracting mass, without integration.

From the expression for V the attraction can be immediately found by the formula of Art. 20. Thus

$$\text{attraction} = -\frac{dV}{dr}$$

$$= \frac{4\pi\rho a^3}{3r^2} + \frac{4\pi\rho a^2}{r^2}\left\{Y_0 + \frac{2a}{3r} Y_1 + \dots \frac{(i+1)\,a^i}{(2i+1)r^i} Y_i + \dots\right\}.$$

PROP. *To find the attraction of a homogeneous body, differing but little from a sphere, on a particle within its mass.*

48. We must in this case expand V in an ascending series of powers of r; and shall therefore take the second of the series of Art. 44. By proceeding as in the last Proposition, we find that the part of V which appertains to the excess over the sphere

$$= \rho \int_{-1}^{1}\int_{0}^{2\pi} \left\{a^2 P_0 + a r P_1 + \dots + \frac{r^i}{a^{i-2}} P_i + \dots\right\} y' d\mu' d\omega',$$

$$\text{or } = 4\pi\rho a^2 \left\{Y_0 + \frac{r}{3a} Y_1 + \dots + \frac{r^i}{(2i+1)\,a^i} Y_i + \dots\right\}.$$

Adding to this the part of V which appertains to the sphere of radius a, viz. $2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2$, for the whole mass,

$$V = 2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2 + 4\pi\rho a^2 \left\{Y_0 + \frac{r}{3a} Y_1 + \dots + \frac{r^i}{(2i+1)\,a^i} Y_i + \dots\right\},$$

and the attraction $= -\dfrac{dV}{dr}$

$$= \frac{4\pi}{3}\rho r - 4\pi\rho a \left\{\frac{1}{3} Y_1 + \frac{2r}{5a} Y_2 + \dots + \frac{i r^{i-1}}{(2i+1)\,a^{i-1}} Y_i + \dots\right\}.$$

49. We can show that by properly choosing the value of (a) and the origin of the radius of the surface we can make Y_0 and Y_1 disappear from the above formulæ.

PROP. *To show that by choosing* a *equal to the radius of the sphere of which the mass equals that of the attracting body we cause Y_0 to vanish, and by taking the centre of gravity of the body as the origin of the radius vector, we cause Y_1 to vanish.*

50. The mass of the body

$$=\rho\int_0^{\mathrm{r}}\int_{-1}^1\int_0^{2\pi} r^2\,dr\,d\mu\,d\omega=\frac{1}{3}\rho\int_{-1}^1\int_0^{2\pi}\mathrm{r}^3\,d\mu\,d\omega,$$

where r is the radius vector of the surface of the body, and $=\mathrm{a}(1+y)$ suppose. Putting this for r, the mass of the body

$$=\text{mass of sphere (rad.} = \mathrm{a}) + \rho\mathrm{a}^3\int_{-1}^1\int_0^{2\pi} y\,d\mu\,d\omega$$

$$=\text{mass of sphere} + \rho\mathrm{a}^3\int_{-1}^1\int_0^{2\pi} Y_0,\ \text{by Art. 29,}$$

$$=\text{mass of sphere} + 4\pi\rho\mathrm{a}^3 Y_0.$$

If then a be taken equal to the radius of the sphere of which the mass equals the mass of the body, $Y_0=0$, as was stated.

51. Again, let $\bar{x}\bar{y}\bar{z}$ be the co-ordinates to the centre of gravity of the body, M its mass: the co-ordinates to the element of which the mass is $-\rho r^2\,dr\,d\mu\,d\omega$ are

$$r\sqrt{1-\mu^2}\cos\omega,\ r\sqrt{1-\mu^2}\sin\omega,\ \text{and } r\mu;$$

$$\therefore M.\bar{x}=\int_0^{\mathrm{r}}\int_{-1}^1\int_0^{2\pi}\rho r^3\sqrt{1-\mu^2}\cos\omega\,dr\,d\mu\,d\omega$$

$$=\frac{1}{4}\int_{-1}^1\int_0^{2\pi}\rho\mathrm{r}^4\sqrt{1-\mu^2}\cos\omega\,d\mu\,d\omega,$$

$$M.\bar{y}=\int_0^{\mathrm{r}}\int_{-1}^1\int_0^{2\pi}\rho r^3\sqrt{1-\mu^2}\sin\omega\,dr\,d\mu\,d\omega$$

$$=\frac{1}{4}\int_{-1}^1\int_0^{2\pi}\rho\mathrm{r}^4\sqrt{1-\mu^2}\sin\omega\,d\mu\,d\omega,$$

$$M.\bar{z}=\int_0^r\int_{-1}^1\int_0^{2\pi}\rho r^3\mu\, dr\, d\mu d\omega=\frac{1}{4}\int_{-1}^1\int_0^{2\pi}\rho \mathrm{r}^4\mu\, d\mu d\omega;$$

putting $\mathrm{r}=\mathrm{a}(1+y)=\mathrm{a}(1+Y_0+Y_1+\ldots+Y_i\ldots)$, and observing that $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$, and μ satisfy Laplace's Equation, and are of the first order, we have by Art. 29,

$$M.\bar{x}=\rho\mathrm{a}^4\int_{-1}^1\int_0^{2\pi}Y_1\sqrt{1-\mu^2}\cos\omega\, d\mu d\omega,$$

$$M.\bar{y}=\rho\mathrm{a}^4\int_{-1}^1\int_0^{2\pi}Y_1\sqrt{1-\mu^2}\sin\omega\, d\mu d\omega,$$

$$M.\bar{z}=\rho\mathrm{a}^4\int_{-1}^1\int_0^{2\pi}Y_1\mu\, d\mu d\omega.$$

But Y_1, being a function of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$ of the first order, is of the form

$$A\sqrt{1-\mu^2}\cos\omega+B\sqrt{1-\mu^2}\sin\omega+C\mu;$$

$$\therefore\ M.\bar{x}=\frac{4}{3}\pi\rho\mathrm{a}^4A,\ M.\bar{y}=\frac{4}{3}\pi\rho\mathrm{a}^4B,\ M.\bar{z}=\frac{4}{3}\pi\rho\mathrm{a}^4C.$$

Hence if we take the origin of co-ordinates at the centre of gravity and therefore $\bar{x}=0, \bar{y}=0, \bar{z}=0$, we have $A=0$, $B=0$, $C=0$, and therefore $Y_1=0$, as stated in the enunciation.

PROP. *To find the attraction of a homogeneous body upon a particle without it; the body consisting of thin strata nearly spherical, homogeneous in themselves, but differing one from another in density.*

52. Let $a'(1+y')$ be the radius of the external surface of any stratum, a' being chosen so that

$$y'=Y_1'+Y_2'+\ldots+Y_i'+\ldots(\text{Art. } 50).$$

Since the strata are supposed not to be similar to one another, y' is a function of a' as well as of μ' and ω'. Let ρ' be the density of the stratum of which the mean radius is a'. Now the value of V for this stratum equals the difference between the values of V for two homogeneous bodies of

the density ρ' and mean radii a' and $a'-da'$. But for the body of which the mean radius is a' (Art. 46)

$$V=\frac{4\pi\rho' a'^3}{3r}+\frac{4\pi\rho' a'^3}{r}\left\{\frac{a'}{3r}Y_1'+\ldots+\frac{a'^i}{(2i+1)r^i}Y_i'+\ldots\right\}.$$

Hence for the stratum of which the external mean radius is a',

$$V=\frac{4\pi\rho' a'^2}{r}da'+\frac{4\pi\rho'}{r}\frac{d}{da'}\left\{\frac{a'^4}{3r}Y_1'+\ldots+\frac{a'^{i+3}}{(2i+1)r^i}Y_i'+\ldots\right\}da',$$

and therefore for the whole body,

$$V=\frac{4\pi}{r}\int_0^a\rho'\left\{a'^2+\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1'+\ldots+\frac{a'^{i+3}}{(2i+1)r^i}Y_i'+\ldots\right)\right\}da'.$$

From which the attraction is easily deduced.

PROP. *To find the attraction of the same body on an internal particle.*

53. Let $r=a(1+y)$ be the radius of the stratum in which the attracted particle lies. Then for the strata within the surface of which the radius is $a(1+y)$, we have

$$V=\frac{4\pi}{r}\int_0^a\rho'\left\{a'^2+\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1'+\ldots+\frac{a'^{i+3}}{(2i+1)r^i}Y_i'+\ldots\right)\right\}da'.$$

But for a stratum external to the particle we have by Art. 48,

$$V=4\pi\rho' a' da'+4\pi\rho'\frac{d}{da'}\left\{\frac{ra'}{3}Y_1'+\ldots+\frac{r}{(2i+1)a'^{i-2}}Y_i'+\ldots\right\}da'.$$

Consequently for the whole body,

$$V=\frac{4\pi}{r}\int_0^a\rho'\left\{a'^2+\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1'+\ldots+\frac{a'^{i+3}}{(2i+)r^i}Y_i'+\ldots\right)\right\}da'$$

$$+4\pi\int_a^a\rho'\left\{a'da'+\frac{d}{da'}\left(\frac{ra'}{3}Y_1'+\ldots+\frac{r^i}{(2i+1)a'^{i-2}}Y_i'+\ldots\right)\right\}da'.$$

From this the attraction is readily obtained by differentiating with respect to r.

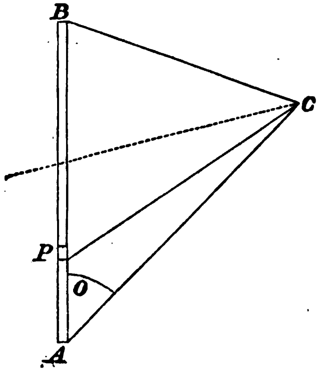
CHAPTER IV.

ATTRACTION OF TABLE-LANDS, MOUNTAINS, OCEANS, &c.

54. The methods which have hitherto been given enable us to find the attraction of the Earth and other bodies of our system considered as a whole. But, taking the Earth as our example, the surface is irregular and does not follow any exact law of form. We ought, therefore, to be able to calculate the effect of these irregularities, and with this view the present Chapter is added to what has gone before. High table-lands may very materially affect the position of the plumb-line in some places. Enormous irregular mountain masses, like the Himalayas, may do the same. Their effect ought, therefore, to be carefully estimated, as the Pendulum and all instruments which are fixed by the Plumb-line or spirit-level must be affected by such irregularities.

Prop. *To find the attraction of a slender prism of matter on a point in the line drawn to one of its extremities.*

55. Let AB be the prism, C the attracted point, P any element of the prism, $AP=r$, M the mass and l the length of the prism, $AC=a$, $BC=b$, $PC=y$, angle $PAC=\theta$.



Then the mass of the element at $P=M\dfrac{dr}{l}$.

Attraction of element at P on $C=M\dfrac{dr}{l}\dfrac{1}{y^2}$.

Ditto in direction CA

$$=M\frac{dr}{l}\frac{1}{y^2}\cos PCA.$$

$$=\frac{Mdr}{l}\frac{a-r\cos\theta}{y^3}, \quad y^2=a^2+r^2-2ar\cos\theta,$$

$$=\frac{Mdr}{l}\cdot\frac{a\sin^2\theta-\cos\theta\,(r-a\cos\theta)}{\{a^2+r^2-2ar\cos\theta)^{\frac{3}{2}}};$$

$\therefore$ attraction of whole prism

$$=\frac{M}{l}\frac{r-a\cos\theta+a\cos\theta}{a\sqrt{a^2+r^2+2ar\cos\theta}}, \text{ from } r=0 \text{ to } r=l,$$

$$=\frac{M}{al}\frac{l}{\sqrt{a^2+l^2-2al\cos\theta}}=\frac{M}{ab}.$$

As this is symmetrical with respect to a and b, it shows that the particle is attracted equally towards the two extremities of the prism; and that therefore the resultant attraction acts in a line bisecting the angle which the prism subtends at the attracted point.

56. A uniform bar of very great length attracts a point not far from its centre with a force varying inversely as the distance from the bar. For let xy be the co-ordinates to the point from the centre measured along the bar and at right angles to it: $2l$ the length of the bar, M its mass. The bar is divided into two parts by y, and they attract the point towards the bar with forces

$$M\frac{l-x}{2l}\frac{1}{y\sqrt{y^2+(l-x)^2}}, \text{ and } M\frac{l+x}{2l}\frac{1}{y\sqrt{y^2+(l+x)^2}}.$$

The sum of these, when l is very large in comparison with x and y, is $M\div yl$, which varies inversely as y.

The following is an approximate illustration of this. The Himalayas resemble a very long prism running W.N.W. and E.S.E. through a point in latitude 33° 30′ in the longitude of Cape Comorin. They attract three places in the meridian of Cape Comorin—viz. Kaliana (lat. 29°30′48″), Kalianpur (24°7′11″), and Damargida (18°3′15″), so as to produce deflections in the plumb-line in the meridian equal to about 28″, 12″, 7″ (see Art 89). The deflections towards the prism or axis of the Himalayas may be taken to bear the same

proportion to each other as those in the meridian. Now the distances of the three stations from the point where the axis crosses the meridian are $3^\circ 39'$, $9^\circ 23'$, $15^\circ 27'$ or $219'$, $563'$, $927'$; in the same proportion are the distances of the stations from the prism or axis. It will be found that the reciprocals of these numbers are not far from being in the same proportion as the deflections. If Kalianpur were removed $20'$ north the comparison would be exact.

57. The following problem may easily be solved by the reader. A is a point on the Earth's surface, taken to be a sphere; AB a vertical column of matter, and AC a column drawn down into the Earth through which there is uniform deficiency of matter, the total deficiency being equal to the mass of the column AB. This column AB in fact may be supposed drawn from the attenuated column AC. Let S be a point on the surface where the horizontal attraction, and therefore the position of the plumb-line, is unaffected by this transfer. S is thus determined. Take Am, as the versine of the arc AS, equal to $AC-AB$, and draw mS perpendicular to AC cutting the surface in S. AB is supposed to be very small indeed compared with the Earth's radius.

PROP. *To find the attraction of a slender pyramid of any form upon a particle at its vertex; and also of a frustum of the pyramid.*

58. Let l be the length of the pyramid, α the area of a transverse section at distance unity from the vertex; r the distance of any section; αr^2 is its area; ρ the density of the matter: then $\alpha r^2 \rho dr$ is the mass of an element of the pyramid, and this divided by r^2 is its attraction;

$$\therefore \text{ attraction of pyramid on vertex} = \int_0^l \alpha\rho dr = \alpha\rho l.$$

If d is the length of any frustum of the pyramid, and $l = l' + d$, then

attraction of pyramid, length l', $= \alpha\rho l'$;

$$\therefore \text{ attraction of frustum} = \alpha\rho d.$$

It is observable that this is quite independent of the distance of the frustum from the vertex; and therefore all portions of the pyramid of equal length, any where selected, attract the vertex equally.

COR. Let the angular *width* of a horizontal pyramid be β and remain constant, while the angular depth varies; and let k be the linear depth of the transverse section of the base; then βlk is the area of the base; and the attraction of the whole pyramid on the vertex $=\rho\beta k$. Hence, all slender pyramids having the same angular width and the same linear depth at the base attract their vertex alike, whatever their lengths be: or, which is the same thing, the angular width being the same the attraction varies as the linear depth of the base, and is independent of the length. Thus, suppose it is required to find the effect of the deficiency of matter in the sea on a place on the sea coast, the shore of which shelves gradually. By dividing the sea into slender horizontal pyramids the attraction of the shelving portion of it can be calculated by knowing only the depth at the extremities of the pyramids without knowing their lengths.

59. By means of this Prop. we may easily find the attraction of a cone on its vertex. For any elementary cone cut out of it will attract the vertex with a force $\alpha\rho l'$, l' being its length, and this resolved along the axis will be $\alpha\rho h$, h being the height of the cone. As α is the transverse section of the elementary cone at a distance unity from the vertex, the sum of all its values for all the elementary cones of which the given cone is made up, it may easily be shown, equals $2\pi(l-h)\div l$, l being the length of a side of the cone. Hence the attraction of the whole cone on the vertex $=2\pi\rho h(l-h)\div l$.

PROP. *To find the attraction of an extensive circular flat plain of given depth or thickness upon a station above its middle point.*

60. Let t be the thickness or depth; h the height of the particle from the nearer surface, c the radius, r the radius of any intermediate elementary annulus of the attracting mass, z its depth. The several elements of this annulus of matter

will attract the particle towards the plane equally. Hence attraction of the particle

$$=\int_0^c\int_0^t \frac{2\pi\rho r(h+z)\,dr\,dz}{\{r^2+(h+z)^2\}^{\frac{3}{2}}} = \int_0^c 2\pi\rho\left\{\text{const.} - \frac{r}{\sqrt{r^2+(h+z)^2}}\right\} dr$$

$$=2\pi\rho\int_0^c\left\{\frac{r}{\sqrt{r^2+h^2}} - \frac{r}{\sqrt{r^2+(h+t)^2}}\right\} dr$$

$$=2\pi\rho\left\{\sqrt{c^2+h^2}-h-\sqrt{c^2+(h+t)^2}+h+t\right\}$$

$$=2\pi\rho t\left(1-\frac{\sqrt{c^2+(h+t)^2}-\sqrt{c^2+h^2}}{t}\right)$$

$$=2\pi\rho t\left(1-\frac{2h+t}{\sqrt{c^2+(h+t)^2}+\sqrt{c^2+h^2}}\right)$$

expanding in powers of $2h+t$

$$=2\pi\rho t\left\{1-\frac{h+\frac{1}{2}t}{\sqrt{c^2+h^2}}+\ldots\right\}.$$

61. If the plain be of infinite extent, the attraction equals $2\pi\rho t$; and this remarkable result is true, that it is independent of the distance from the plain.

The same will be the case if the height of the station above the middle of the attracting mass below, that is, $h+\frac{1}{2}t$, be so small that it may be neglected in comparison with the distance of the station from the furthest limit of the plain. The formula $2\pi\rho t$ will give the result within 1 – 10th of the true value, if c is not less than $10h+5t$; and within 1 – 100th if c is not less than $100h+50t$. Thus if $h=1$ mile, and $t=10$ miles, c must not be less than 60 and 600 miles in the two cases.

62. The result of this Proposition when the plain is unlimited in extent, viz. that the vertical attraction depends only on the thickness of the plain and not on the height above it, might have been foreseen from the result in the previous Proposition regarding the attraction of the frustum of a pyramid. Conceive an infinite number of slender pyramids to be drawn from the station intersecting the attracting plain; they will cut out of it an equal number of frusta, and the cosines of the

angles they make with the perpendicular to the plain will be the thickness divided by the lengths of the frusta. But the attractions of the frusta are proportional to their lengths, and independent of the distance from the attracted point: (see Art. 58). Hence the resultant attraction of the whole will depend solely upon the thickness or depth of matter constituting the plain, and not at all upon the height of the station above the plain.

The rationale of this curious result, that the attraction of the plain, when infinite in extent, is the same whatever the height above the plain, is this. Conceive the apparatus of slender geometrical pyramids of equal angular width, mentioned in the last paragraph, to be moved bodily upwards with the station. The several frusta of these pyramids, which they cut out of the plain, will shift and increase in volume (except the last) and also in distance from the station, but their *lengths* will be as before. Hence their vertical attractions at the station will be unchanged. The increase of volume of the frusta is made, it will be seen, by drawing upon the inexhaustible store of the *last* frustum, which is infinite in extent.

The same result is approximately true for a plain of a limited but very great extent compared with the height of the station.

63. The last Prop. enables us to suggest an explanation (as far as the facts are at present before us) of a remarkable example of local horizontal attraction which has of late years been detected near Moscow. At Moscow Observatory there is a northern deflection of the plumb-line of about 8″. The neighbourhood has been surveyed, and the following are the results: see *Monthly Notices of the Royal Astronomical Society*, No. 6, April 10, 1863. "The amount of disturbance near Moscow, in relation to the distance of a place from the central line [running about E.N.E. and W.S.W.], is nearly as follows:

At distance	0	English miles,	disturbance	=	0.
,,	2·5	,,	,,	=	2″·22.
,,	8	,,	,,	=	7·80.
,,	13	,,	,,	=	5·15.
,,	18	,,	,,	=	2·10.
,,	23	,,	,,	=	0.

Its magnitude is sensibly the same, but with an opposite sign at equal distances north and south of the central line." Moscow stands on the north line of maxima. It will be observed that the disturbance increases in passing north or south from the central line (as far as the observations show) through 8 miles, and in proportion to the distance; and through the next 15 miles it diminishes again at about half the rate of its former increase. Suppose there is below the surface a broad dyke 16 miles wide and of great depth and extent E.N.E. and S.W.S., immediately south of Moscow, throughout which a deficiency of matter exists. This will cause a horizontal attraction which can be estimated by Art. 61, which may account for the phenomenon. For in passing S.S.E. from Moscow through s miles to another station, the effect on that station of the part of the dyke passed over will be counteracted by that of the next s miles, and the effective part will have become only $16-2s$ miles wide, and the nearest part of it will be s miles off. If we pass from Moscow (which stands just on the edge of the supposed dyke) towards the N.N.W. through s miles, the whole dyke is effective, and its distance is s, as before. If the extent of the dyke were infinite and the surface of the ground a plane throughout, then (by Art. 61) the attraction in moving away north from the dyke would remain constant. But its extent must be limited; and therefore, on moving north from Moscow, the attraction will decrease. On moving southward the effect will also decrease from this same cause, viz. the dyke not being infinite in extent. But up to the central line it will decrease more rapidly from another cause: the width of the effective part of the dyke varies as $16-2s$, or as $8-s$, that is, as the distance from the central line. While this will lessen the rate of decrease mentioned above as arising from the dyke being limited, it will greatly promote the decrease by lessening the effective cause of disturbance, and, at the central line itself, annihilating it altogether. This accords with the observed facts; viz. that in passing from the line of maxima to the central line the decrease of disturbance varies nearly as the distance from the central line; and in passing away in the opposite direction, the decrease is more slow.

On the supposition of the extent of the deficiency being very great, let n be the ratio of deficiency to the mean density

of the earth. The deflection being 8″ at the edge (at Moscow), the horizontal attraction must be $g \tan 8'' = 0{\cdot}00003879\,g$. This (by Art. 61) $= \pi\rho \times 16$ miles $= 0{\cdot}003\,gn$, the radius of the earth being taken 4000 miles,

$$\therefore\ n = 0{\cdot}01293 = 1\text{-}77\text{th nearly}.$$

Hence a deficiency of only 1-77th of the mean density or 1-39th of the density of the rock, prevailing throughout the dyke, will explain the phenomenon.

PROP. *To correct for elevation above the sea-level.*

64. When the pendulum is used as a measure of gravity, the time of vibration will be altered by any change in the density of the material beneath it, as well as by its elevation above the sea-level. If ρ is the density of rock, taken to be half the mean density of the earth, $g = \frac{8}{3}\pi\rho a$. Hence the attraction of an extensive plain $= 2\pi\rho t = \frac{3}{4}\frac{t}{a}g$. Suppose, owing to geological changes of level, a continent is lifted up above the mean surface of the earth through a space t. Then gravity at a station on the continent will be diminished from this cause by the amount

$$g\left\{1 - \left(1 + \frac{t}{a}\right)^{-2}\right\} = \frac{2t}{a}g \text{ nearly}.$$

But the attraction of the underlying mass of thickness t must be taken into account. Hence the real diminution of gravity by the upheaval will be $\frac{5}{4}\frac{t}{a}g$. The ratio of this to the correction for increase of distance $= 0{\cdot}625$. This is commonly called Dr Young's correction*.

If the station be at the height h above the level of the continent, then the diminution

$$= \left(\frac{2h}{a} + \frac{5}{4}\frac{t}{a}\right)g.$$

* Dr Young takes the ratio of the density of the surface to the mean density to be 5 : 11. In this case the correction would be $58 \div 88 = 0{\cdot}66$. See *Phil. Trans.* 1819, p. 93.

This correction will depend upon the kind of rock of which the continent is made, whether of a dense or light description. Thus also for a station at sea, like St Helena, the correction would be different for a similar height above a continent, as sea water is only half the density of rock.

65. The above is the method of correction which has been hitherto generally used in reducing pendulum observations to the sea-level in order to compare them together. Of late years much more attention has been paid to the effect of local attraction than formerly: and extensive pendulum experiments have been made along the Great Arc of Meridian and at other stations in India, as well as in other countries, to assist in detecting variations of density below. In 1868 the author, feeling that some more accurate means of correcting for the effect of variations of the earth's superficial mass were required, calculated and printed in a pamphlet new formulæ, which he reproduces here in the following fourteen Articles.

PROP. *To find the attraction of a cylinder or flat circular disk on a point at either extremity of its axis.*

66. Let a be the radius of each end, h its length, z the distance from one end of a transverse section of thickness dz, r the radius of a circle on this section around the axis. Then all the elements of the ring of matter at this circle will attract the given point alike, and the total attraction

$$= \int_0^a \int_0^h \frac{2\pi\rho r z dr dz}{(z^2+r^2)^{\frac{3}{2}}} = \int_0^a 2\pi\rho\left(1 - \frac{r}{\sqrt{h^2+r^2}}\right) dr$$
$$= 2\pi\rho\left(a + h - \sqrt{a^2+h^2}\right).$$

67. This is easily reduced to numbers for any particular case. If the calculation is required for a great many cases, the following method may be pursued. The formula may be written

$$\text{Attraction} = 2\pi\rho a\left(1 + \frac{h}{a} - \sqrt{1 + \frac{h^2}{a^2}}\right),$$

$$\text{or} = 2\pi\rho h\left(\frac{a}{h} + 1 - \sqrt{\frac{a^2}{h^2} + 1}\right);$$

and the calculation in both cases depends upon the value of m

$$= 1 + n - \sqrt{1 + n^2}.$$

If then we form a table of values of m for a regular progression of values of n, all greater than 1, and not differing much from each other, so that we may easily interpolate for intermediate values of n, we shall have the means ready at hand for finding the attraction at once for any pair of values of a and h. For we have but to find the numerical ratio of h to a, or of a to h, taking that which is greater than 1, search for it among the values of n in the table, or interpolate for it, and find the corresponding value of m. This multiplied by $2\pi\rho a$, or by $2\pi\rho h$, as the case may be, gives the attraction.

PROP. *To find the attraction of a spherical cap of matter of uniform thickness on the mid-points of its outer and of its inner surfaces.*

68. The cap is part of a spherical shell, its outer rim being generated by the revolution of a radius of the larger bounding spherical surface around the radius through the attracted points, at a constant angle.

Let t be the thickness of the cap; c and r the distances of the attracted point and of any particle of the cap from the centre of the spheres; θ the angle r makes with c; $z = c - r$; c chord $\theta = u$; c vers $\theta = v$.

First, suppose the attracted point outside.

The attraction of an elementary ring of matter round the point reckoned positive towards the centre

$$= 2\pi r \sin\theta \, . \, dz \, . \, r d\theta \, . \, \rho \frac{c - r\cos\theta}{(c^2 + r^2 - 2cr\cos\theta)^{\frac{3}{2}}}$$

$$= \frac{2\pi\rho r^2}{c^2} \frac{d}{d\theta}\left(\frac{r - c\cos\theta}{\sqrt{c^2 + r^2 - 2cr\cos\theta}}\right) d\theta\, dz$$

$$= \frac{2\pi\rho}{c^2}(c - z)^2 \frac{d}{d\theta}\left(\frac{2c\sin^2\frac{1}{2}\theta - z}{\sqrt{z^2 + 4c(c - z)\sin^2\frac{1}{2}\theta}}\right) d\theta\, dz.$$

Integrating from $\theta = 0$ to $\theta = \theta$, and then putting

$$2c\sin\tfrac{1}{2}\theta = u, \text{ and } u^2 = 2cv,$$

total attraction of the cap

$$=\frac{2\pi\rho}{c^2}\int_0^t\left\{(c-z)^2+\frac{(v-z)(c-z)^2}{\sqrt{z^2+2(c-z)v}}\right\}dz\ \ldots\ldots\ldots\ldots (1)$$

$$=\frac{2\pi\rho}{c^2}\left\{\frac{c^3-(c-t)^3}{3}+\int_0^t\frac{(v-z)(c-z)^2\,dz}{\sqrt{z^2-2vz+2cv}}\right\}.$$

The integral, by parts, becomes

$$-(c-z)^2\sqrt{z^2-2vz+2cv}-2\int(c-z)\sqrt{z^2-2vz+2cv}\,dz$$

$$=-(c-z)^2\sqrt{z^2-2vz+2cv}+\frac{2}{3}(z^2-2vz+2cv)^{\frac{3}{2}}$$

$$+2(v-c)\int\sqrt{z^2-2vz+2cv}\,dz$$

$$=-(c-z)^2(z^2-2vz+2cv)^{\frac{1}{2}}+\frac{2}{3}(z^2-2vz+2cv)^{\frac{3}{2}}$$

$$-(v-c)\{(v-z)\sqrt{z^2-2vz+2cv}\}-(v^2-2cv)\log_e(v-z+\sqrt{z^2-2vz+2cv})$$

$$=-\left\{\frac{z^2}{3}-\left(c-\frac{v}{3}\right)z-c^2-\frac{7}{3}cv+v^2\right\}(z^2-2vz+2cv)^{\frac{1}{2}}$$

$$+(v-c)(v^2-2cv)\log_e(v-z+\sqrt{z^2-2vz+2cv}).$$

Putting this for the integral, and replacing $2cv$ by u^2, and taking the limits, Vertical Attraction of the cap

$$=\frac{2\pi\rho c}{3}\left\{1-\left(1-\frac{t}{c}\right)^3+\frac{3u}{c}-\frac{7u^3}{2c^3}+\frac{3u^5}{4c^5}\right.$$

$$-\left\{\frac{3u}{c}-\frac{7u^3}{2c^3}+\frac{3u^5}{4c^5}-\left(\frac{3u}{c}-\frac{u^3}{2c^3}\right)\frac{t}{c}+\frac{u}{c}\frac{t^2}{c^2}\right\}\sqrt{\frac{t^2+u^2}{u^2}-\frac{t}{c}}$$

$$\left.+3\left(\frac{u^2}{2c^2}-1\right)\left(\frac{u^4}{4c^3}-\frac{u^2}{c^2}\right)\log_e\frac{\frac{u}{2c}-\frac{t}{u}+\sqrt{\frac{t^2+u^2}{u^2}-\frac{t}{c}}}{\frac{u}{2c}+1}\right\}\ldots\ldots(2).$$

This is the exact expression.

Secondly, suppose the attracted point inside. The formula (2) requires in this case some modification. In the first place, when the limits of θ are taken to obtain (1) and θ is put $=0$, the radical in the denominator is $-z$, and not z as before. This will change the sign of the first term $(c-z)^2$ in (1), and will change the signs of the first and second terms within the brackets in (2). Again, the limits of integration with regard to z must be taken from $z=-t$ to $z=0$, which is the same as putting $-t$ for t, and also changing the sign of every term of (2). Making these changes, and still estimating the attraction positive towards the centre, vertical attraction of the cap

$$= \frac{2\pi\rho c}{3}\left\{1-\left(1+\frac{t}{c}\right)^3-\frac{3u}{c}+\frac{7u^3}{2c^3}-\frac{3u^5}{4c^5}\right.$$

$$+\left\{\frac{3u}{c}-\frac{7u^3}{2c^3}+\frac{3u^5}{4c^5}+\left(\frac{3u}{c}-\frac{u^3}{2c^3}\right)\frac{t}{c}+\frac{u}{c}\frac{t^2}{c^2}\right\}\sqrt{\frac{t^2+u^2}{u^2}+\frac{t}{c}}$$

$$\left.-3\left(\frac{u^2}{2c^2}-1\right)\left(\frac{u^4}{4c^4}-\frac{u^2}{c^2}\right)\log_e\frac{\frac{u}{2c}+\frac{t}{u}+\sqrt{\frac{t^2+u^2}{u^2}+\frac{t}{c}}}{\frac{u}{2c}+1}\right\} \quad \ldots\ldots(3).$$

As these formulæ are to be applied to find the vertical attraction of the spherical portions of the earth, it may be here stated that as these attractions will be always small quantities the earth may be regarded as a sphere, and c taken equal to the mean radius 3956 miles, as the height of any station above the sea-level will always be small, generally less than 1 mile.

PROP. *To reduce the above formulæ for use by approximation.*

69. For even the highest mountains in the world the ratio $t \div c$ will not be greater than about $1 \div 800$ and its square may be neglected. Expand, then, formula (2) in powers of $t \div c$ and neglect its square; observing that as c^2 occurs in the denominator of every term of the coefficient of the log, we may neglect t^2 everywhere in the log itself. Hence, Vertical Attraction

$$=2\pi\rho\left\{t+u-\frac{7u^3}{6c^2}+\frac{u^5}{4c^4}-\sqrt{t^2+u^2}+\frac{7u^3}{6c^2}-\frac{u^5}{4c^4}+\frac{ut}{c}-\frac{u^3t}{6c^3}\right.$$

$$\left.+\frac{ut}{2c}-\frac{7u^3t}{12c^3}+\frac{u^5t}{8c^5}+\left(\frac{u^2}{2c^2}-1\right)\left(\frac{u^4}{4c^3}-\frac{u^2}{c}\right)\log_e\left(1-\frac{t}{u}\right)\right\}$$

$$=2\pi\rho\left(u+t-\sqrt{u^2+t^2}+\frac{ut}{2c}\right).$$

If we take the density of the surface to be half the mean density of the earth, and g be gravity, then this formula becomes, Vertical Attraction

$$=\frac{3g}{4c}\left(u+t-\sqrt{u^2+t^2}+\frac{ut}{2c}\right) \text{ ....} \quad (4).$$

If formula (3) be in like manner expanded it gives Vertical Attraction

$$=-\frac{3g}{4c}\left(u+t-\sqrt{u^2+t^2}-\frac{ut}{2c}\right) \text{ ....} \quad (5).$$

These formulæ we will now apply to various uses.

Prop. *To extend Dr Young's correction for elevation above the sea-level.*

70. When ut is sufficiently small to allow of the last term in (4) and (5) being neglected, the formulæ become those for the attraction of a cylinder or a flat circular disk: see Art. 66.

Suppose the value of gravity is observed on the highest point of a spherical cap of matter lying with its lower surface on the sea-level at a depth t, and we wish to reduce the observed value of gravity to the sea-level, for both the height t and the attraction of the cap. Let L be the correction to be added on account of change of level. Then

$$L=g\left\{\left(\frac{c}{c-t}\right)^2-1\right\}=\frac{2t}{c}g.$$

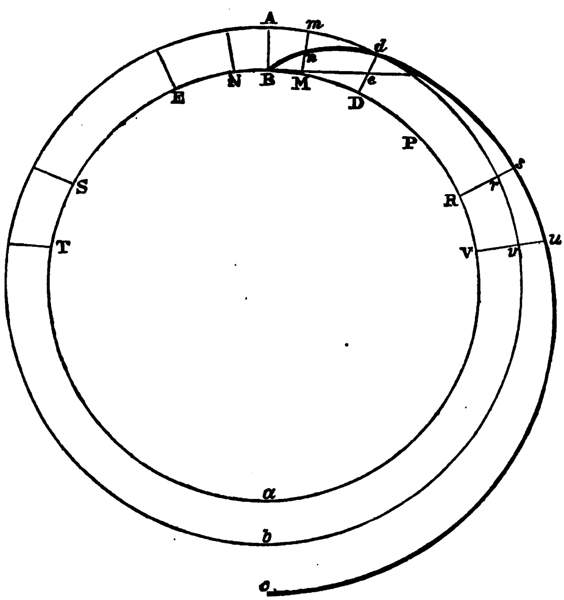
Hence, eliminating g by means of this, the whole correction for both change of level and attraction of the cap

$$= L - \text{vertical attraction of cap, (4),}$$

$$= \frac{5}{8} L \left\{ 1 + \frac{3}{5} \frac{\sqrt{u^2 + t^2} - u}{t} - \frac{3u}{10c} \right\}.$$

The first term of this is the correction commonly known as Dr Young's correction: see Art. 64. It supposes that the attracting mass is a plain, and a plain of infinite extent. The formula we have thus deduced introduces two corrections of this. The second term, as the formula is here arranged, limits the infinite extent of the plain; and the third term corrects for the earth's curvature, as the limited circular plain or disk is curved down into a spherical cap. This is evident; for when c is infinite the third term vanishes, and when u is infinite the second does.

71. Dr Young's formula is by far the largest part of the whole correction except in extreme cases. The accompanying diagram is intended to exhibit graphically the amount of vertical attraction as given by the complete formula.



Suppose the inner circle is the sea-level, and the outer circle the upper surface of the spherical cap. A the station, $AB=t$, u the chord from B to any point on the sea-level,

$$z=u+t-\sqrt{u^2+t^2}+\frac{ut}{2c}.$$

Then the vertical attraction at A varies as z for different values of u. The various values of z are erected vertically on the sea-level so as to trace out the curve in the diagram. The vertical ordinate, for example, Rs at R shows the relative magnitude of the vertical attraction at A of the cap SAR, stretching as far as R and S. At the antipodes $u=2c$, $z=2t$, neglecting extremely small quantities. Hence in the diagram $ac=2t$. When the attraction is half this, $z=t$;

$$\therefore\ t=u+t-\sqrt{u^2+t^2}+\frac{ut}{2c},$$

$$\therefore\ u^2+t^2=u^2+\frac{u^2t}{c}+\frac{u^2t^2}{4c^2},$$

$$\therefore\ u^2=ct\div\left(1+\frac{t}{4c}\right)=ct \text{ very nearly.}$$

Hence if a tangent be drawn at B, and the point D be so chosen that the tangent bisects the thickness Dd in e, the vertical attraction of the cap EAD at A is half the attraction of the whole shell, and the curve crosses the outer circle at d. If we put $u=BM=\frac{3}{4}t$, $z=\frac{1}{2}t=Mn$. When u is extremely small, the formula for z becomes

$$z=u-\frac{u^2}{2t},$$

which shows that at B the curve goes off at 45° with the horizontal. Hence the general form of the curve will be at once seen. It rises most suddenly at first, and after crossing the level of A at the comparatively near point d, rises very gradually indeed till it attains twice the height of d at the antipodes. This shows how very gradually the vertical attraction of a cap increases as its radius increases after passing d.

Thus for any two caps SAR and TAV the attractions are measured by Rs and Vu, and therefore the attraction of the zone $RVST$ is only the difference between rs and vu, a minute quantity.

PROP. *To show that for* 76 *miles radius from the station the cap may be taken to be a flat disk.*

72. We will consider that the term depending on the curvature may be neglected when it equals 1-100th of the whole: or

$$\frac{ut}{2c\,(u+t-\sqrt{u^2+t^2})}=\frac{1}{100}\,;$$

$$\therefore\ u+t+\sqrt{u^2+t^2}=\frac{c}{25}\,,$$

$$u^2+t^2=(u+t)^2-\frac{2c}{25}(u+t)+\frac{c^2}{625}\,,$$

$$u=\left(\frac{2ct}{25}-\frac{c^2}{625}\right)\div\left(2t-\frac{2c}{25}\right),\ c=3956,\ t=5$$

$$=\frac{c}{25}\,\frac{50t-c}{50t-2c}=\frac{c}{25}\,\frac{3706}{7662}$$

$$=76 \text{ miles.}$$

This is for the extreme case of t being 5 miles.

PROP. *To find how high the station may be lifted up above the cap without sensibly affecting the amount of vertical attraction of the cap upon it.*

73. Let h be the height of the station above the cap, and u the extent of the cap. Then the vertical attraction on the station equals the difference of the effects of two caps of thicknesses $t+h$ and h

$$=\frac{3g}{4c}\left\{u+t+h-\sqrt{u^2+(t+h)^2}+\frac{u\,(t+h)}{2c}\right.$$
$$\left.-u-h+\sqrt{u^2+h^2}-\frac{uh}{2c}\right\}$$

$$= \frac{3g}{4c}\left\{t - \frac{t^2 + 2th}{2u} + \frac{ut}{2c}\right\}$$

$$= \frac{3gt}{4c}\left\{1 + \frac{u}{2c} - \frac{t}{2u} - \frac{h}{u}\right\}.$$

If h does not exceed 1-100th of u the term in h may be neglected, and the vertical attraction of the cap may be considered independent of the height of the station.

This is analogous to the result explained in Art. 62 for a flat plain of great though finite extent. The cap may be regarded as a *level* plain, that is, a plain of which the upper surface is parallel to the sea-level.

From this it follows, that if the cap is distributed into horizontal layers and h is not more (say) than 1-100th of u for any one of the layers, and the total mass is unaltered, then the vertical attraction at the station is unaltered by any change of arrangement of the layers.

74. As an example of the use of these calculations we will solve the following problem, which will be found of use hereafter. Suppose a small circular island of radius a is standing on a cylindrical base in the midst of an extensive sea of uniform depth h. Suppose the crust or sea-bed to have an excess of matter through the thickness t exactly equal to the defect of matter in the sea when compared with the density of rock. Suppose also that the density of the island is that of rock, all through the depth $h + t$, so that everywhere on the island and the sea, the amount of matter in a vertical column of depth $h + t$ is the same. It is required to find how much the vertical attraction at the centre of the island differs from what it would be if the sea were rock and the crust throughout were of that density. We shall suppose h, t, and u so chosen that the Prop. of the last Article applies. Hence if there were no island, and the whole were a sea of uniform depth and the crust as described above throughout, the vertical attraction at the point which was the centre of the island is the ordinary gravity for that latitude. But in our problem this is modified by the introduction of the circular island standing on a cylinder: and to find what that modification is, we must subtract from the vertical attraction of a cylinder of

rock of length $t+h$ on the centre of the upper end, the vertical attraction of a cylinder of sea-water of density 0·363 times that of rock and of length h, and of a cylinder of density $(1+0·637h \div t)$ times that of rock and length t, the upper end being at a depth h below the station. The fourth powers of $a \div t$ and $a \div (t+h)$ may be neglected. Hence, by formula (4) of Art. 69, we have excess of vertical attraction

$$= \frac{3g}{4c}\left\{a+t+h-\sqrt{a^2+(t+h)^2}+\frac{a(t+h)}{2c}\right.$$

$$-\left(a+h-\sqrt{a^2+h^2}+\frac{ah}{2c}\right)0·363$$

$$-\left(a+t+h-\sqrt{a^2+(t+h)^2}+\frac{a(t+h)}{2c}\right.$$

$$\left.\left.-a-h+\sqrt{a^2+h^2}-\frac{ah}{2c}\right)\left(1+\frac{0·637h}{t}\right)\right\}$$

$$= \frac{3g}{4c}\left\{\left(a+h-\sqrt{a^2+h^2}+\frac{ah}{2c}\right)0·637\,\frac{t+h}{t}\right.$$

$$\left.-\left(a+t+h-\sqrt{a^2+(t+h)^2}+\frac{a(t+h)}{2c}\right)\frac{0·637h}{t}\right\}$$

$$= \frac{g}{8280}\left[(a+h-\sqrt{a^2+h^2})\,\frac{h+t}{t}-\left\{a-\frac{a^2}{2(t+h)}\right\}\frac{h}{t}\right].$$

Suppose $a=5$ miles, $h=1$, $t=80$. Then

$$\text{excess of gravity} = 0·0001028g.$$

By this quantity is the vertical attraction at the centre of the island increased, beyond what it would be if the surface and crust were all rock, in consequence of the different distribution of the matter. The rationale of this, on the hypothesis introduced into the problem, is obvious. The matter around the island which has by hypothesis been transferred from the sea to the crust below is in a new position, where its effect in producing vertical attraction is greater than before.

This example has been worked out in order to explain how a theory regarding the constitution of the earth's crust, to be advanced in a future part of this book, accounts for the fact that gravity on the little island Minicoy, 250 miles west of Cape Comorin, is greater, although out at sea, than at Cape Comorin itself in the ratio of 1·0000894. Minicoy Island varies from 4 to 6 miles in diameter. It is nowhere more than 30 feet above the sea-level. Beyond 3 miles from the shore the depth is more than 1-3rd of a mile: and the sea-bottom of course shelves considerably beyond this. If we allow for this, and round off the right angles at the top and bottom of our cylindrical island, we may consider the measures we have assumed to be a fair representation of Minicoy. We shall revert to this case hereafter.

PROP. *To show that if the radius of the cap is more than 50 times the thickness, the vertical attraction varies sensibly as the thickness.*

75. By (4) of Art. 69 we have by expansion vertical attraction $= \frac{3gt}{4c}\left(1 - \frac{t}{2u} + \frac{u}{2c} + \dots\right)$. If $t \div 2u = 1 \div 100$, that is, $u = 50t$, the second term may be neglected, and the vertical attraction varies sensibly as the thickness, for the same value of u.

The same result would be obtained by using formula (5).

COR. If a zone of matter lies beyond this limit, the vertical attraction of each of the two caps of which the zone is the difference varies as the thickness. Hence the attraction of the zone, and therefore of any vertical prism of the zone, varies as the thickness. Hence, if the prism be divided into any number of equal parts by horizontal planes, the vertical attraction of all of these parts will be the same. Hence any portion of matter distant from the station more than 50 times its height above the sea-level may be transferred upwards or downwards through a space reaching from the sea-level to the station-level, and also to an equal height above the station-level without sensibly altering its vertical attraction on the station.

Also it is evident that the vertical attraction will not be altered by transferring the portion of matter in azimuth to any extent around the station. Hence the matter in a whole irregular zone, being more than 50 times its greatest height and depth above or below the station-level, may be re-arranged by levelling down in any way, so long as it continues over the zone, without affecting the vertical attraction: or, if we take the average height for the actual varying height of the matter on the zone the vertical attraction will not be affected. This is of use when approximating to the attraction of an undulating country on a station.

PROP. *To divide the cap or sea-level into zones.*

76. Let u and w be the chords of the angular distances from the station of the bounding circles of any zone drawn around the station on the sphere which represents the sea-level. Except very near the station—which part will be otherwise treated—the ratio $t \div u$ is so small that its fourth power may be neglected. Then the formula (4) of Art. 69 gives for a cap of thickness t lying below the station level, vertical attraction

$$= \frac{3g}{4c}\left(t + \frac{ut}{2c} - \frac{t^2}{2u}\right),$$

and therefore for the mass over the zone, vertical attraction

$$= \frac{3g}{4c}\,\frac{w-u}{2c}\left(t + \frac{t^2 c}{uw}\right).$$

Hence also if h be the height of the station above the sea-level, the vertical attraction of a mass on the zone up to the level of the station

$$= \frac{3g}{4c}\,\frac{w-u}{2c}\left(h + \frac{h^2 c}{uw}\right).$$

Taking the difference of these, and putting the height of the superficial matter on the zone above the sea-level, i.e. $h - t, = k$, Vertical Attraction of mass standing on the zone

$$= \frac{3g}{4c}\,\frac{w-u}{2c}\,k\left(1 + \frac{2h-k}{c}\,\frac{c^2}{uw}\right) \ldots\ldots\ldots\ldots\ldots (1).$$

If we use the formula (5) of Art. 69, in which case the superficial mass rises above the station-level, vertical attraction

$$=\frac{3g}{4c}\frac{w-u}{2c}\left(t-\frac{t^2c}{uw}\right).$$

This must be added to the attraction of the mass between the sea-level and the station-level. Observing that in this case $k=h+t$, we have, Vertical Attraction of mass standing on the zone

$$=\frac{3g}{4c}\frac{w-u}{2c}\left\{h+t+\frac{(h^2-t^2)c}{uw}\right\}$$

$$=\frac{3g}{4c}\frac{w-u}{2c}k\left(1+\frac{2h-k}{c}\frac{c^2}{uw}\right),$$

precisely the same formula as before.

The formula is also true when applied to parts covered by the ocean. Let as before h be the height of the station above the sea-level, but k the depth of the ocean (supposed uniform under the zone). The ratio of the density of sea-water to that of rock, considered half the mean density of the earth or 5·56, $=0·363$. Then $h+k$ and k are the heights of the station and the surface of the attracting ocean above the level of the ocean-bed; and therefore by formula (1), Vertical Attraction of the ocean under the zone

$$=\frac{3g}{4c}\frac{w-u}{2c}0·363\,k\left\{1+\frac{2(h+k)-k}{c}\frac{c^2}{uw}\right\},$$

and therefore the effect of the *deficiency* of density in the ocean below that of rock

$$=-\frac{3g}{4c}\frac{w-u}{2c}0·637\,k\left(1+\frac{2h+k}{c}\frac{c^2}{uw}\right),$$

which is precisely the same as formula (1), $-k$ being put for k because it is measured below the sea-level; and the density being that of the deficiency of attracting matter. Hence the formula (1) is true in all cases.

The law of dissection of the sea-level into zones which we shall choose is that which will simplify this formula as much as possible. We shall make the difference of chords to the

bounding circles the same in all the zones, that is, $w-u=b$, a constant for all the zones. Suppose that the whole sphere of the sea-level is divided into n parts around the station according to the above law, viz. a circular part immediately in its neighbourhood, $n-2$ zones, and a circular part around the antipodes, and let $u_1\, u_2\, u_3 \ldots u_n$ be the chords to the successive bounding circles. Then

$$u_1 = b,\ u_2 = u_1 + b = 2b, \ldots\ u_n = nb,\ = \text{also } 2c;$$

$$\therefore \frac{u_1}{2c} = \frac{1}{n},\ \frac{u_2}{2c} = \frac{2}{n}, \ldots \frac{u_n}{2c} = \frac{n}{n} = 1.$$

According to this law of division, Vertical Attraction of a mass standing on the rth zone after the central portion

$$= \frac{3g}{4c}\frac{k}{n}\left\{1 + \frac{2h-k}{4c}\frac{n^2}{r(r+1)}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

a very simple expression.

The value of n should be so chosen that the zones may not be too large to allow the average height of the masses standing on the zones to represent the masses fairly. In making an estimate of this we may bear in mind, that matter may be always transferred *in azimuth* round the station without altering its effect on the vertical attraction at the station. An application of this principle may frequently in practice assist in getting a good average for the mass without contracting the zone too much. See also Art. 75, Cor.

Any zone can easily be subdivided into smaller zones if necessary. Also any zone may be divided into four-sided compartments by great circles, so drawn through the station as to divide it into portions, the average heights of which will better represent the mass, if it be irregular, than the mean height of the whole would.

Prop. *In the event of the upper surface of the attracting mass not being horizontal, to find the effect of taking the average height to represent the mass.*

77. If we differentiate the formulæ (4), (5) in Art. 69 with respect to u we have vertical attraction of mass corresponding to an elementary zone

$$=\frac{3g}{4c}\left(1-\frac{u}{\sqrt{u^2+t^2}}+\frac{t}{2c}\right)du,$$

$$\text{or } =-\frac{3g}{4c}\left(1-\frac{u}{\sqrt{u^2+t^2}}-\frac{t}{2c}\right)du,$$

according as the matter lies below or above the station-level.

I. Let us take the circular portion of the superficial mass on the sea-level immediately around the station. Suppose its upper surface shelves down gradually so that $t=\alpha u$, where α is a small quantity. Then the first of the above formulæ enables us to find the excess of vertical attraction, which we introduce by considering the mass level with the station and not sloping down. This excess

$$=\frac{3g}{4c}\int\left(1-\frac{1}{\sqrt{1+\alpha^2}}+\frac{\alpha u}{2c}\right)du$$

$$=\frac{3g}{8c}\left(\alpha^2 u+\frac{\alpha u^2}{2c}\right)=\frac{3g}{8c}\left(\frac{t^2}{u}+\frac{tu}{2c}\right),$$

t and u here having their values at the extremity of the circular mass. Suppose $u=c\div 100$, also $c=3956$. Hence vertical attraction

$$=\frac{t^2+0{\cdot}2t}{420000}g.$$

The variations of gravity as determined by the most accurate pendulum experiments are generally reduced to seven places of decimals. To make this equal to $0{\cdot}0000001\,g$, or to produce a unit in the seventh place of decimals in the ratio to gravity, we must have

$$t^2+0{\cdot}2t=0{\cdot}042,$$

$$t=-0{\cdot}1+\sqrt{0{\cdot}01+0{\cdot}042}=0{\cdot}052,$$

$$=-0{\cdot}1+0{\cdot}23=0{\cdot}13=2\text{-15ths mile}.$$

To make it equal to $0{\cdot}0000010\,g$, or to produce a unit in the sixth place of decimals, we must have

$$t=-0{\cdot}1+\sqrt{0{\cdot}01+0{\cdot}42}=0{\cdot}43$$

$$=-0{\cdot}1+0{\cdot}66=0{\cdot}56=4\text{-7ths mile}.$$

The radial extent of the circular central portion is about 40 miles. Hence if the *average* inclination of the ground, taken *all around* the station, be as much as 2-15ths of a mile in the 40 miles, the error in considering the surface horizontal will be only one unit in the seventh place of decimals of the ratio to gravity, a quantity which is evanescent. And if the incline be as much as 4-7ths of a mile, an error of only one unit in the sixth place of decimals will be incurred.

It will be easily seen that if the ground rises from the station, and the second formula is therefore used, the rise necessary to produce a given error must even be greater than the incline in the last Article.

II. We will now consider one of the zones, and suppose that the attracting mass lies wholly below the station-level, and that the upper surface slopes gradually in passing on all sides from the station so that t, the depth of the upper surface below the station-level, $=\alpha(u-v)+p$, v being the mean of the two chords to the bounding circles of the zone, p the distance of the mass below the station-level corresponding to v, which may therefore be regarded as the mean distance of the mass below the station-level: u_r and u_{r+1} the limiting chords, and $(u_{r+1}-u_r)^3$ being small enough to be neglected in comparison of $(u_{r+1}+u_r)^3$: also α is small, and its fourth power will be neglected, and p will be considered of the same order of smallness as α. Hence Vertical Attraction of a mass between the station-level and the upper surface of our actual mass would

$$=\frac{3g}{4c}\int\left(1-\frac{u}{\sqrt{u^2+\{\alpha(u-v)+p\}^2}}+\frac{\alpha(u-v)+p}{2c}\right)du$$

$$=\frac{3g}{4c}\left\{u-\frac{\sqrt{u^2+\{\alpha(u-v)+p\}^2}}{1+\alpha^2}+\frac{\alpha u^2-2(\alpha v-p)u}{4c}\right.$$

$$\left.+\frac{\alpha p-\alpha^2 v}{(1+\alpha^2)^{\frac{3}{2}}}\log_e\frac{\sqrt{1-\alpha^2}u+\alpha p-\alpha^2 v+\sqrt{u^2+(\alpha u+p-\alpha v)^2}}{\text{constant}}\right\},$$

neglecting the powers and products of α and p to the fourth order,

$$=\frac{3g}{4c}\left\{u-u(1-\alpha^2)-\frac{(\alpha u-\alpha v+p)^2}{2u}+\frac{\alpha u^2-2(\alpha v-p)u}{4c}\right.$$

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$$+ (\alpha p - \alpha^2 v) \log_e (u \times \text{const.})\Big\}.$$

Taking the limits and observing that

$$\frac{1}{u_r u_{r+1}} = \frac{4}{(u_r + u_{r+1})^2 - (u_{r+1} - u_r)^2} = \frac{1}{v^4} + \frac{(u_{r+1} - u_r)^2}{4v^4},$$

$$\log_e \frac{u_{r+1}}{u_r} = \log\left(1 + \frac{u_{r+1} - u_r}{u_{r+1} + u_r}\right) - \log\left(1 - \frac{u_{r+1} - u_r}{u_{r+1} + u_r}\right)$$

$$= \frac{u_{r+1} - u_r}{v},$$

Vertical Attraction

$$= \frac{3g}{8c} \cdot u_{r+1} - u_r) \left\{\alpha^2 + \frac{(\alpha v - p)^2}{v^2} + \frac{\alpha v}{c} - \frac{\alpha v - p}{c} + \frac{2\alpha p - 2\alpha^2 v}{v}\right\}$$

$$= \frac{3g}{8c} (u_{r+1} - u_r) \left(\frac{p^2}{v^2} + \frac{p}{c}\right).$$

This is independent of α, although we have not neglected powers lower than the 4th, reckoning p as of the same order as α. This result shows that no serious error will be incurred by taking the average height of the matter on the zone above the sea-level to represent the actual mass, unless there are any extensive and abrupt variations in the upper surface.

The same will be true, and in a somewhat greater degree, if the mass rises above the station-level. Also, in both cases, the slope may increase or decrease (*i.e.* α may be positive or negative) in passing from the station.

This shows that if a limited tract of country undulates very much, but has no great and extensive hollows or elevations, the whole may be treated as a table-land, that is, as land the upper surface of which is parallel to the sea-level.

78. The process, then, to be followed for obtaining at any station the Vertical Attraction of the superficial mass lying above the sea-level is this. Lay down the zones on a map according to the law deduced above. Write down the height (h) of the station and the average heights (k) of the mass on the successive zones above the sea-level. If the heights in any one zone vary considerably, it must be divided

into compartments by great circles through the station, and each to be treated separately. Substitute the heights, for the central part, in the formula for a cylinder in Art. 66, and for the zones in the formula (2) of Art. 76. Add the results and the sum is the vertical attraction at the station of the whole superficial mass standing above the sea-level.

PROP. *Suppose that beneath a cap lying on the sea-level there is a uniform attenuation of matter equal to that of the cap, running down to a depth* m *times the thickness of the cap: to find the Resultant Vertical Attraction of the cap and the attenuation at the highest point of the cap.*

79. Let h be the thickness of the cap. Then by Art. 69

$$\text{Vert. Attraction of cap} = \frac{3g}{4c}\left(u + h - \sqrt{u^2 + h^2} + \frac{uh}{2c}\right).$$

The effect of the attenuation is equal to the difference of the effects of two caps, of $1 \div m$th the density of the cap, running down to depths h and $h + mh$ below the upper level of the cap: and therefore

$$= -\frac{3g}{4cm}\left\{u + (1+m)h - \sqrt{u^2 + (1+m)^2 h^2} + \frac{u(1+m)h}{2c} - \left(u + h - \sqrt{u^2 + h^2} + \frac{uh}{2c}\right)\right\}$$

$$= -\frac{3g}{4c}\left\{h + \frac{\sqrt{u^2 + h^2} - \sqrt{u^2 + (1+m)^2 h^2}}{m} + \frac{uh}{2c}\right\},$$

adding this to the attraction of the cap, Resultant Vertical Attraction of the cap

$$= \frac{3g}{4c}\left\{u - \frac{m+1}{m}\sqrt{u^2 + h^2} + \frac{1}{m}\sqrt{u^2 + (1+m)^2 h^2}\right\}.$$

If we may neglect the fourth power of $(1+m)h \div u$, this

$$= \frac{3g}{4c}\,\frac{(1+m)h^2}{2u}.$$

PROP. *To find the Resultant Vertical Attraction of the mass on a zone, under the same circumstances.*

80. If k be the height of the mass on the zone, then by Art. 76, formula (2), Vertical Attraction of the mass on the zone

$$=\frac{3g}{4cn}\left\{k-\frac{k^2-2kh}{4c}\,\frac{n^2}{r(r+1)}\right\}.$$

The effect of the attenuation is equal to the difference of the effects of two similar masses, of $1\div m$th the density of the given mass, reaching down through spaces $h+mk$ and h below the station-level. As $h+mk$ may be too large to allow our omitting its fourth power, we cannot use the above formula to find this effect; but must revert to the original formula from which it is derived as an approximation. Thus from formula (4) of Art. 69, we derive the vertical attraction of a mass of thickness t below the station-level and standing on the rth zone, and of the density of rock (see Art. 76),

$$=\frac{3g}{4c}(w-u)\left(1-\frac{\sqrt{w^2+t^2}-\sqrt{u^2+t^2}}{w-u}+\frac{t}{2c}\right)$$

$$=\frac{3g}{2n}\left\{1-\frac{\sqrt{4(r+1)^2c^2+n^2t^2}-\sqrt{4r^2c^2+n^2t^2}}{2c}+\frac{t}{2c}\right\}:$$

The effect of the attenuation, therefore,

$$=\frac{3g}{4cnm}\left\{\sqrt{4(r+1)^2c^2+n^2(h+mk)^2}-\sqrt{4r^2c^2+n^2(h+mk)^2}\right.$$
$$\left.-\sqrt{4(r+1)^2c^2+n^2h^2}+\sqrt{4r^2c^2+n^2h^2}-mk\right\}$$

$$=\frac{3g}{4cnm}\left\{\sqrt{4(r+1)^2c^2+n^2(h+mk)^2}-\sqrt{4r^2c^2+n^2(h+mk)^2}\right.$$
$$\left.-2c+\frac{n^2h^2}{4r(r+1)c}-mk\right\}.$$

FIRST: suppose that the ratio of the depth of the attenuation to the mean radius of the zone is such, that its

fourth power may be neglected. Then the effect of the attenuation

$$=\frac{3g}{4cn}\left[-\frac{n^2}{4r(r+1)mc}\{(h+mk)^2-h^2\}-k\right].$$

Adding this to the vertical attraction of the mass on the zone, at the beginning of this Article, we have

Resultant Vertical Attraction of mass on rth zone

$$=-\frac{3gn}{16c^2}\,\frac{(m+1)k^2}{r(r+1)}.$$

SECONDLY: suppose that $h+mk$ is too large for the fourth power to be neglected. We will introduce a subsidiary angle ϕ.

Put $\dfrac{n(h+mk)}{(2r+1)c}=\tan\phi.$

Then adding the attraction of the zone and the effect of the attenuation, and substituting, we have

Resultant Vertical Attraction for the zone

$$=\frac{3g}{4mn}\Big[\sqrt{4(r+1)^2+(2r+1)^2\tan^2\phi}-\sqrt{4r^2+(2r+1)^2\tan^2\phi}$$
$$-2-\frac{n^2}{4c^2r(r+1)}\{m(k^2-2kh)-h^2\}\Big].$$

The pair of radicals in this expression

$$=\sqrt{(2r+1)^2\sec^2\phi+1+2(2r+1)}-\sqrt{(2r+1)^2\sec^2\phi+1-2(2r+1)}$$

$$=\frac{2(2r+1)}{\{(2r+1)^2\sec^2\phi+1\}^{\frac{1}{2}}}+\frac{(2r+1)^3}{\{(2r+1)^2\sec^2\phi+1\}^{\frac{5}{2}}}\text{ by expansion}$$

$$=2\cos\phi\left\{1-\frac{\cos^2\phi}{2(2r+1)^2}\right\}+\frac{\cos^5\phi}{(2r+1)^2}\left\{1-\frac{5\cos^2\phi}{2(2r+1)^2}\right\}\text{ nearly}$$

$$=2\cos\phi-\frac{\cos^3\phi-\cos^5\phi}{(2r+1)^2}\text{ nearly}$$

$$=2\cos\phi-\frac{2\cos\phi-\cos3\phi-\cos5\phi}{16(2r+1)^2};$$

$\therefore$ Resultant Vertical Attraction for the zone

$$= \frac{3g}{2mn}\Big[-1 + \cos\phi - \frac{2\cos\phi - \cos 3\phi - \cos 5\phi}{32\,(2r+1)^2}$$
$$- \frac{n^2}{8c^2 r\,(r+1)}\{m\,(k^2 - 2kh) - h^2\}\Big].$$

Calling the part within the bracket R; and supposing only an angular part β of the zone is covered with matter, at the altitude k,

$$\text{Resultant Vertical Attraction} = \frac{3g}{2mn}\,\frac{\beta}{360}\,R.$$

The expression for R may be somewhat simplified for zones sufficiently distant. For when ϕ is sufficiently small to allow us to neglect its fourth power, the tangent gives

$$\phi^2 = \frac{n^2}{c^2}\left(\frac{h + mk}{2r+1}\right)^2,$$

and the first part of R depending on ϕ becomes by expansion

$$-\frac{\phi^2}{2}\left\{1 + \frac{1}{(2r+1)^2}\right\} + \frac{\phi^4}{24}\left\{1 + \frac{22}{(2r+1)^2}\right\}.$$

Neglecting ϕ^4 and substituting for ϕ^2, this becomes

$$-\frac{n^2}{2c^2}\left(\frac{h+mk}{2r+1}\right)^2\left\{1 + \frac{1}{(2r+1)^2}\right\},$$

and $$R = -\frac{n^2}{2c^2}\left[\frac{(h+mk)^2}{(2r+1)^2}\left\{1 + \frac{1}{(2r+1)^2}\right\} + \frac{m\,(k^2 - 2kh) - h^2}{4r\,(r+1)}\right].$$

In order to ascertain for what zones this simpler formula for R may be used, it is to be remembered that in the final result decimals are to be retained to the seventh place in the ratio of vertical local attraction to gravity. Hence

$$3\beta R \div 720mn$$

must be calculated to seven places of decimals. Hence a quantity in it as small as 0·0000001 must be retained, or a

quantity in R as small as $0{\cdot}000024mn \div \beta$. Hence the neglected term

$$\frac{\phi^4}{24}\left\{1+\frac{22}{(2r+1)^2}\right\} \text{ or } \frac{n^4}{24c^4}\left(\frac{h+mk}{2r+1}\right)^4\left\{1+\frac{22}{(2r+1)^2}\right\} \text{ must be}$$

$$<\frac{24mn}{10^6\beta} \text{ or } 2r+1 > \frac{n\,(h+mk)}{0{\cdot}153}\left[\frac{\beta}{mn}\left\{1+\frac{22}{(2r+1)^2}\right\}\right]^{\frac{1}{4}}.$$

When numerical values are given to the quantities involved, it is always easy to find the least value of r which satisfies this condition: that value of r shows the first zone for which the second form of R may be used.

The author has made use of these formulæ in an interesting problem regarding the constitution of the earth's crust, published in the *Philosophical Transactions* for 1871.

PROP. *To find the attraction of a rectangular mass, of small elevation compared with its length and breadth, upon a point lying in the plane of one of its larger sides.*

81. Let the attracted point be the origin of co-ordinates; the tabular base of the mass the plane of xy, the axes of x and y parallel to its long edges, the axis of z being parallel to its thickness. Let $x'y'z'$ be the co-ordinates to any point of the mass: xy co-ordinates to the nearest angle, XY to the furthest angle, H the height of the mass; ρ the density, supposed the same throughout.

Then $\rho dx'\,dy'\,dz'$ is the mass of the element; and the height being small, we may suppose the element projected on the plane of xy. Hence the whole attraction parallel to x

$$=\rho\int_x^X\int_y^Y\int_0^H\frac{x'\,dx'\,dy'\,dz'}{\{x'^2+y'^2\}^{\frac{3}{2}}}=\rho H\int_x^X\int_y^Y\frac{x'\,dx'\,dy'}{\{x'^2+y'^2\}^{\frac{3}{2}}}$$

$$=\rho H\int_x^X\left\{\frac{Y}{\sqrt{x'^2+Y^2}}-\frac{y}{\sqrt{x'^2+y^2}}\right\}\frac{dx'}{x'}$$

$$=\rho H\log_e\left\{\frac{\sqrt{1+\frac{Y^2}{x^2}}+\frac{Y}{x}}{\sqrt{1+\frac{Y^2}{X^2}}+\frac{Y}{X}}\cdot\frac{\sqrt{1+\frac{y^2}{X^2}}+\frac{y}{X}}{\sqrt{1+\frac{y^2}{x^2}}+\frac{y}{x}}\right\}.$$

To simplify the formula put

$$\frac{Y}{x}=\tan\theta_1,\quad \frac{y}{X}=\tan\theta_2,\quad \frac{Y}{X}=\tan\theta_3,\quad \frac{y}{x}=\tan\theta_4;$$

$$\therefore \sqrt{1+\frac{Y^2}{x^2}}+\frac{Y}{x}=\frac{1+\sin\theta_1}{\cos\theta_1}=\tan\theta\,(45^\circ+\tfrac{1}{2}\theta_1),$$

and so of the rest. Hence, since 0·434 is the modulus of common logarithms,

$$\text{attraction}=\frac{\rho H}{0{\cdot}434}\Big\{\log\tan(45^\circ+\tfrac{1}{2}\theta_1)+\log\tan(45^\circ+\tfrac{1}{2}\theta_2)$$
$$-\log\tan(45^\circ+\tfrac{1}{2}\theta_3)-\log\tan(45^\circ+\tfrac{1}{2}\theta_4)\Big\},$$

which gives a remarkably simple rule for finding the attraction parallel to x: that parallel to y can be found in like manner.

It is easy to show, that if the density be half the mean density of the earth, that is, about the same as granite, g be gravity, the radius of the earth $=3956$ miles, and H be expressed in feet, the coefficient above $=gH\tan\left(\frac{1''}{368}\right)$. Hence, since the tangent of deflection of the plumb-line caused by the attraction equals, by the parallelogram of forces, the ratio of the attraction to gravity, and the angle is very small,

Deflection of plumb-line caused by the Tabular Mass parallel to the axis of x

$$=\frac{1''}{368}H\Big\{\log\tan(45^\circ+\tfrac{1}{2}\theta_1)+\log\tan(45^\circ+\tfrac{1}{2}\theta_2)$$
$$-\log\tan(45^\circ+\tfrac{1}{2}\theta_3)-\log\tan(45^\circ+\tfrac{1}{2}\theta_4)\Big\}.$$

It is evident that the Tabular Mass may be partly below and partly above the plane of xy, so long as the height or depth is not so great that its square may not be neglected in comparison with the square of the distance from the attracted point. In this case H is the sum of the height and depth, above and below the plane of xy.

Ex. 1. The co-ordinates to the nearest and furthest angles of a tabular block of rock measured from the attracted point

are 3 and -16, 40 and 30 miles, and the height of the mass from bottom to top is 628 feet. Show that the deflection of the plumb-line at the station taken as origin, and parallel to the shorter side of the parallelogram, $= 3''\cdot18$.

Ex. 2. A table-land 1610 feet high, commencing at a distance of 20 miles north from Takal K'hera, near the Great Arc of Meridian in India, runs 80 miles further north, and 60 miles to the east and 60 to the west. Find the deviation of the plumb-line at that station. It is about $5''$; so considerable as to have induced Sir G. Everest to abandon that place as a principal station.

82. In cases where the attracting mass is near, it is necessary to cut it up into prisms and calculate the effect of each separately and add the results. Examples of this are seen in the celebrated case of Schehallien, and more recently in the calculation of thé deflection at Arthur's Seat, Edinburgh, by Sir H. James, Superintendent of the Ordnance Survey. See *Philosophical Transactions* for 1856, p. 591.

83. The irregular character of the surface of the Earth, consisting of mountain and valley and ocean, may in some instances have a sensible effect, by presenting an excess or deficiency of attracting matter, upon the position of the plumb-line, in such a way as to derange delicate survey operations. Hindostan affords a remarkable example of this, as the most extensive and the highest mountain-ground in the world lies to the north of that continent, and an unbroken expanse of ocean stretches south down to the south pole. Both these causes, by opposite effects, would make the plumb-line hang somewhat north of the true vertical.

In the following Propositions a method is laid down for calculating the attraction of an irregular superficial stratum of the Earth's surface, and making it depend altogether upon the contour of the surface. The method pursued is this: A law of geometric dissection of the surface is discovered which divides it into a number of four-sided spaces, such that if the height of the attracting mass were the same in them all, they would all attract the given station exactly to the same amount, whether far or near. In this case it would be necessary only

to calculate for one space; then count the number of spaces in each lune in the country under consideration, and the final result is easily attained. The country being supposed irregular, the heights in the spaces will not be all alike. The principle, therefore, should be stated thus, that the attractions of the masses on the several compartments are in proportion to their mean heights. These mean heights are known by knowing the contour of the country.

PROP. *To discover a Law of Dissection of the surface of the earth into compartments, so that the attraction of the masses of matter standing on them, upon a given station in the horizontal direction, shall be exactly proportional to the mean heights of the masses, be they far or near.*

84. Suppose a number of great circles to be drawn from the station in question to the antipodes, making any angle β, each with the next, thus dividing the earth's surface (which we may in this calculation suppose to be a sphere, without incurring any sensible error) into a number of lunes. Then, with the station as centre, describe on the surface a number of circles, at distances the law of which it is our object now to determine, dividing the whole into a number of four-sided compartments.

We will begin by calculating the attraction of a mass of matter, standing on one of these compartments, at a uniform height throughout, upon the station in a horizontal direction. Let α and $\alpha+\phi$ be the angular distances from the station of the two circles bounding this compartment; h the height of the mass; θ the angular distance along the surface of an elementary vertical prism of the mass; a the radius of the earth; ψ the angle which the plane of θ makes with the plane of the mid-line of the lune, and in which latter plane the resultant attraction evidently acts. The area of the base of the prism $= a^2 \sin\theta d\psi d\theta$.

Since the height of the prism (h) is supposed very small, the distances of its two extremities from the station may be taken to be the same, and $=2a\sin\frac{1}{2}\theta$. Its attraction along the chord of θ

$$=\frac{\rho a^2 h \sin\theta \, d\theta \, d\psi}{4a^2 \sin^2 \frac{1}{2}\theta}.$$

Attraction along the tangent to $\theta = \dfrac{\rho h \sin\theta \, d\theta \, d\psi}{4 \sin^2 \frac{1}{2}\theta} \cos \frac{1}{2}\theta$;

$\therefore$ attraction along the tangent to the mid-line of the lune

$$= \frac{\rho h \cos^2 \frac{1}{2}\theta \, d\theta \, d\psi}{2 \sin \frac{1}{2}\theta} \cos \psi;$$

$\therefore$ attraction of the whole mass

$$= \frac{\rho h}{2} \int_{\alpha}^{\alpha+\phi} \int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \frac{\cos^2 \frac{1}{2}\theta}{\sin \frac{1}{2}\theta} d\theta \,.\, \cos\psi d\psi = \rho h \sin \tfrac{1}{2}\beta \int_{\alpha}^{\alpha+\phi} \frac{\cos^2 \frac{1}{2}\theta}{\sin \frac{1}{2}\theta} d\theta$$

$$= 2\rho h \sin \tfrac{1}{2}\beta \left\{ \log_e \frac{\tan \frac{1}{4}(\alpha+\phi)}{\tan \frac{1}{4}\alpha} + \cos \tfrac{1}{2}(\alpha+\phi) - \cos \tfrac{1}{2}\alpha \right\}$$

$$= 2\rho h \sin \tfrac{1}{2}\beta \left\{ \log_e \frac{\sin(\frac{1}{2}\alpha + \frac{1}{4}\phi) + \sin \frac{1}{4}\phi}{\sin(\frac{1}{2}\alpha + \frac{1}{4}\phi) - \sin \frac{1}{4}\phi} - 2 \sin(\tfrac{1}{2}\alpha + \tfrac{1}{4}\phi) \sin \tfrac{1}{4}\phi \right\}$$

$$= 4\rho h \sin \tfrac{1}{2}\beta \sin \tfrac{1}{4}\phi \left\{ \frac{1}{\sin(\frac{1}{2}\alpha + \frac{1}{4}\phi)} - \sin(\tfrac{1}{2}\alpha + \tfrac{1}{4}\phi) \right\}$$

$$= 4\rho h \sin \tfrac{1}{2}\beta \frac{\sin \frac{1}{4}\phi \cos^2(\frac{1}{2}\alpha + \frac{1}{4}\phi)}{\sin(\frac{1}{2}\alpha + \frac{1}{4}\phi)},$$

neglecting only the cube and higher powers of $\sin \frac{1}{4}\phi$.

The law of dissection we shall choose will simplify this; for we are to assume such a relation between ϕ and α that the expression in ϕ may be constant, in order to make the attraction the same for all compartments in which h is the same or varying as h where the heights of the masses standing on the compartments are different. As the value of the constant to which we equal the function of α and ϕ is quite arbitrary, we will assume it such that when α and ϕ are small, ϕ shall $= \frac{1}{10}\alpha$. In this case it $= \dfrac{\frac{1}{40}\alpha}{\frac{1}{2}\alpha + \frac{1}{40}\alpha} = \dfrac{1}{21}$.

Hence $$\frac{\sin \frac{1}{4}\phi \cos^2(\frac{1}{2}\alpha + \frac{1}{4}\phi)}{\sin(\frac{1}{2}\alpha + \frac{1}{4}\phi)} = \frac{1}{21} \quad \text{..............(1)},$$

defines the Law of Dissection.

The attraction of the mass standing on the compartment, in consequence,

$$=\frac{4}{21}\rho h \sin\tfrac{1}{2}\beta;$$

an exceedingly simple expression. We may obtain it in terms of gravity as follows. Let ρ the density be the same as that of the mountain Schehallien, viz. 2·75; the mean density, according to Mr Baily's repetition of the Cavendish experiment, being 5·66; g gravity, $a = 4000$ miles.

$$\text{Now } g = \frac{4\pi}{3}a \times \text{mean density} = \frac{4\pi}{3}a\frac{566}{275}\rho;$$

$\therefore$ attraction of mass on any compartment

$$=\frac{4}{21}\frac{3}{4\pi}\frac{275}{566}\frac{h}{a}\sin\tfrac{1}{2}\beta.\, g = 0{\cdot}000005523h\sin\tfrac{1}{2}\beta.g,$$

h being expressed in parts of a mile.

Since $0{\cdot}000005523 = \tan(1''{\cdot}1392)$;

$\therefore$ deflection of the plumb-line caused by this attraction

$$= 1''{\cdot}1392h\sin\tfrac{1}{2}\beta \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

85. The method of using this formula is as follows. When the numerical values of the successive pairs of α and ϕ are determined by the solution of equation (1) giving the law of dissection, lay them and the lunes down on a map of the country the attraction of which is to be found. It will thus be covered with compartments. After examining the map, write down the average heights of the masses standing on all the several compartments of any one lune; add them together, multiply the sum by $1''{\cdot}1392\sin\frac{1}{2}\beta$, and the equation (2) shows that we have the deflection caused by the mass on the whole lune in the vertical plane of its middle line. Multiply by the cosine and then the sine of the azimuth of that middle line, and we have the deflections in the meridian and the prime-vertical. The same being done for all the lunes, and the results added, we have the effects in meridian and prime-vertical produced by the whole country under consideration.

PROP. *To calculate the dimensions of the successive compartments from the law of dissection.*

86. For this purpose we should solve the equation of last Proposition, viz.

$$\frac{\sin \frac{1}{4}\phi \cos^2 (\frac{1}{2}\alpha + \frac{1}{4}\phi)}{\sin (\frac{1}{2}\alpha + \frac{1}{4}\phi)} = \frac{1}{21} \quad\text{..................(1).}$$

But this cannot be done. We must therefore approximate, which will equally well suit our purpose. In order to afford a test of the values we arrive at, the equation may be written under the following form:

$$\log \sin \tfrac{1}{4}\phi^{\circ} = 18\cdot6777807 + \log \sin (\tfrac{1}{2}\alpha + \tfrac{1}{4}\phi)$$
$$- 2 \log \cos (\tfrac{1}{2}\alpha + \tfrac{1}{4}\phi)\text{...........(3).}$$

Equation (1) can be solved by expansion so long as α and ϕ are not too large. It gives

$$\phi = \frac{4}{21}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)\left\{1 - \frac{1}{6}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2 + \left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2\right\}$$

$$= \frac{2}{21}\left(\alpha + \frac{\phi}{2}\right)\left\{1 + \frac{5}{6}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2\right\};$$

$$\therefore \frac{\alpha}{\phi} + \frac{1}{2} = \frac{21}{2}\left\{1 - \frac{5}{6}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2\right\},$$

$$\frac{\alpha}{\phi} + 1 = 11 - \frac{35}{16}(\alpha + \phi - \tfrac{1}{2}\phi)^2,$$

$$\frac{\phi}{\alpha + \phi} = \frac{1}{11}\left\{1 + \frac{35}{176}(\alpha + \phi - \tfrac{1}{2}\phi)^2\right\}$$

$$= \frac{1}{11}\left\{1 + \frac{35}{176}\left(\frac{21}{22}\right)^2(\alpha + \phi)^2\right\} = \frac{1}{11}\{1 + 0\cdot1812\,(\alpha + \phi)^2\},$$

$$= \frac{1}{11}\{1 + 0\cdot000055\,(\alpha + \phi)^2\} \quad\text{.......................... (4),}$$

α and ϕ being expressed in degrees.

Let $\alpha_1\alpha_2\alpha_3 \ldots \phi_1\phi_2\phi_3 \ldots$ be the successive values of α and ϕ for the several compartments of a lune, beginning with the antipodes. These are connected by the following relations:

$$\alpha_1 + \phi_1 = 180^0, \quad \alpha_2 + \phi_2 = \alpha_1, \quad \alpha_3 + \phi_3 = \alpha_2, \quad \&c. \ \ldots\ldots(5).$$

For the first, equation (1) gives

$$21 \sin^3 \tfrac{1}{4}\phi_1 = \cos \tfrac{1}{4}\phi_1, \text{ or } \cot^3 \tfrac{1}{4}\phi_1 + \cot \tfrac{1}{4}\phi_1 - 21 = 0,$$

$$\therefore \cot \tfrac{1}{4}\phi_1 = 2{\cdot}6379, \text{ or } \phi_1 = 83^0\ 2', \ \therefore \ \alpha_1 = 96^0\ 58'.$$

For the second, $\frac{1}{2}(\alpha_2 + \phi_2) = \frac{1}{2}\alpha_1 = 48^0\ 29'$. Putting this in equation (3) we must by trial find the value of ϕ_2 which satisfies it; and so of $\phi_3 \ldots$ This process brings out the series of values of ϕ_2 and α_2, ϕ_3 and α_3, &c. as far as the 22nd, gathered together in the following Table:

No. of Compart.	Values of ϕ	Values of α	No. of Compart.	Values of ϕ	Values of α	No. of Compart.	Values of ϕ	Values of α
1	83° 2′	96° 58′	18	1° 29′	14° 39′	35	0° 17′	2° 55′
2	15 12	81 46	19	1 21	13 18	36	0 16	2 39
3	10 54	70 52	20	1 13	12 5	37	0 14	2 25
4	8 34	62 18	21	1 6	10 59	38	0 13	2 12
5	7 3	55 15	22	1 0	9 59	39	0 12	2 0
6	5 58	49 17	23	0 54	9 5	40	0 11	1 49
7	5 8	44 9	24	0 50	8 15	41	0 10	1 39
8	4 29	39 40	25	0 45	7 30	42	0 9	1 30
9	3 56	35 44	26	0 41	6 49	43	0 8	1 22
10	3 29	32 15	27	0 36	6 13	44	0 7	1 15
11	3 6	29 9	28	0 34	5 39	45	0 7	1 8
12	2 47	26 22	29	0 31	5 8	46	0 6	1 2
13	2 30	23 52	30	0 28	4 40	47	0 6	0 56
14	2 15	21 37	31	0 25	4 15	48	0 5	0 51
15	2 1	19 36	32	0 23	3 52	49	0 5	0 46
16	1 49	17 47	33	0 21	3 31	50	0 4	0 42
17	1 39	16 8	34	0 19	3 12	&c.	&c.	&c.

After the 22nd each of the remaining values of ϕ is obtained by dividing the next preceding values of α by 11, as the small term in equation (4) then becomes insignificant, and the succeeding value of α is easily deduced by means of the formulæ (5). The Table may be carried on to any extent, the only restriction on its use being that the height of the

mass on any compartment must not be so great relatively to its distance from the station that the square of the ratio cannot be neglected.

87. The following Table will be found useful in Art. 90.

Distances of Mid-points of Compartments (counting them from the antipodes) from the station.

1	138° 29′	12	27° 46′	23	9° 32′	34	3° 21′	45	1° 14′
2	89 22	13	25 7	24	8 40	35	3 3	46	1 5
3	76 19	14	22 44	25	7 52	36	2 47	47	0 59
4	66 35	15	20 36	26	7 9	37	2 32	48	0 54
5	58 46	16	18 42	27	6 31	38	2 18	49	0 49
6	52 16	17	16 58	28	5 56	39	2 6	50	0 44
7	46 43	18	15 24	29	5 23	40	1 54	51	0 42
8	41 55	19	13 58	30	4 54	41	1 44	52	0 40
9	37 42	20	12 41	31	4 27	42	1 35	53	0 38
10	34 0	21	11 32	32	4 3	43	1 26	54	0 35
11	30 42	22	10 29	33	4 31	44	1 19	55	0 32

88. COR. The relative effect of the same or an equal and similar mass, situated on different parts of the earth's surface, is easily obtained as follows.

As the effects of the compartments into which any lune is divided are all the same, the height of the mass standing on them being the same, the effect of a given mass standing on any area will vary inversely as the area of the particular compartment in which it is situated. Now if α and $\alpha+\phi$ be the distances of the nearer and further sides of any compartment, and β be the width of the lune, the area of the compartment $=\beta\{\cos\alpha-\cos(\alpha+\phi)\}$. Hence the relative attraction of the same mass in different situations will vary inversely as

$$\{\cos\alpha-\cos(\alpha+\phi)\}.$$

For example; the centre of the Island of Australia is about 36° and 63° from Singapoor and Calcutta; it stands therefore, with reference to those places, on the 9th and 4th compartments, reckoning from their antipodes, and the ratio of the horizontal attractions of the Island on those places

$$=\frac{\cos 62^\circ 18'-\cos 70^\circ 52'}{\cos 35^\circ 44'-\cos 39^\circ 40'}=\frac{0.46484-0.32777}{0.81174-0.76977}$$

$$=\frac{0.13707}{0.04197}=3.266.$$

89. The formulæ above deduced may be applied to find the effect on the plumb-line of any mountain-region, or hollow (as in the case of the ocean), so long as the angle subtended at the station by any vertical line in it is such as to allow its square to be neglected.

Ex. 1. In the *Philosophical Transactions* for 1855 (p. 85) and 1859 (p. 770) the author has applied these principles to find the effect of the Himalayas and the mountain-region beyond them on the plumb-line in India, and has found that the meridian deflection caused in the northern station of the Great Arc of Meridian (lat. 29° 30′ 48″, and long. 77° 42′) is nearly 28″, as far as the data regarding the contour of the mass have been ascertained; and that the astronomical amplitudes between that and the next principal station (lat. 24° 7′ 11″), and between that and the third (lat. 18° 3′ 15″), are diminished by the quantities 15″.9 and 5″.3. He has also shown that the meridian deflection at points between the first and third stations varies very nearly inversely as the distance from a point in the meridian in latitude 33° 30′.

General Chodzko states that at Tiflis, Douchet, Wladikawkas, Alexandrowskaja, and Mosdok, which are severally 70, 35, 35, 55, 70 miles from the central line of the Caucasus, the deflections are (taking that at Tiflis to be zero) 25″.1, − 28″.6, − 12″.0, − 5″.6 North. (See *Monthly Notices of Astron. Soc.* April, 1862.)

Ex. 2. The effect of the deficiency of matter in the Ocean south of Hindostan down to the south pole is also calculated (*Phil. Trans.* 1859, p. 790) by the author, upon an assumed but not improbable law of the depth, and found to produce a meridian deflection northwards at the three stations of the Indian Arc of 6″, 9″, 10″.5 respectively; and 19″.7 at Cape Comorin. The deflections at Karachi, and a point half way between Cape Comorin and Karachi, arising from this cause, are shown to be 10″ and 13″.8.

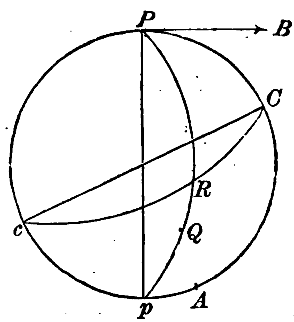
It is not difficult to show from the last three, that the horizontal attraction northwards, at points along the west coast of India, arising from deficiency of matter in the ocean, may be approximately represented by the formula

$$(0.000095556839 - 0.000002836162\lambda + 0.000000004072\lambda^2)\, g,$$

in which λ is the difference of latitude of the station and Karachi, expressed in degrees and parts of a degree. (*Phil. Trans.* 1859, p. 793.)

The formulæ may be applied also to obtain the attraction of thin sections of the earth's surface of a *regular* form which the Integral Calculus does not enable us to calculate. The following is an instance.

PROP. *To find the horizontal attraction of a thin hemi-spherico-spheroidal meniscus of matter lying on the earth's surface upon a point on the surface.*



90. In the diagram let P be the point attracted, at first not on the meniscus but beyond it. CAc the meniscus, A its pole. Suppose the meniscus divided into lunes from P to p, and each lune into compartments. RQp is the mid-line of that portion of one of these lunes which lies on the meniscus, and Q is the mid-point of one of its compartments. Let

$$PQ = \alpha,\ PR = \lambda,\ AP = \theta,\ RPC = \delta:$$

then λ is given by the equation

$$\cot\lambda = -\tan\theta\cos\delta.$$

Let n be the number of compartments between R and p. The numerical value of λ will, by using the Table of Art. 87, determine the value of n.

When P is 180° from the pole of the meniscus, the horizontal attraction will evidently be zero. We will take P successively at 150°, 135°, 120°, and 90° from the pole of the meniscus. The values of λ and n calculated from the above formula are as follows:

δ	Values of θ. 150°		135°		120°		90°	
	λ	n	λ	n	λ	n	λ	n
15°	60° 51′,	4	46° 0′,	6	30° 52′,	10	180°	The whole lune.
45	67 48,	3	54 44,	5	39 14,	8	180	
75	81 30,	2	75 29,	3	66 21,	4	180	
105	98 30,	1	104 31,	1	113 39,	0	180	
135	112 12,	1	125 16,	0	140 46,	0	180	
165	119 9,	0	134 0,	0	149 8,	0	180	

If t be the thickness of the shell at Q and h at the pole A, then by the equation to an ellipse of small ellipticity, we have

$$t = h \cos^2 AQ$$
$$= h (\cos\theta \cos\alpha + \sin\theta \sin\alpha \cos\delta)^2.$$

The resultant horizontal attraction of any lune must be resolved along the tangent PB at P, and will by Art. 84 be

$$\frac{4}{21}\rho \sin 15^\circ \cos\delta\, \Sigma\,.\, t = 0.0493\rho \cos\delta\, \Sigma\,.\, t$$

$$= 0.0493h\rho \cos\delta\Sigma\,.\,(\cos\theta\cos\alpha + \sin\theta \sin\alpha \cos\delta)^2$$

$$= 0.02465h\rho \cos\delta\Sigma.\{2\cos^2\theta - (\cos^2\theta - \sin^2\theta\cos^2\delta)(1-\cos 2\alpha) + \sin 2\theta \cos\delta \sin 2\alpha\}$$

$$= 0.02465h\rho \cos\delta\{2n\cos^2\theta - (\cos^2\theta - \sin^2\theta\cos^2\delta)\,\Sigma\,(1-\cos 2\alpha) + \sin 2\theta\cos\delta\,\Sigma\sin 2\alpha\}.$$

Any one of the values of θ being taken, the Table above shows how many (n) compartments there will be for each lune; and then the corresponding values of α, or the distances of the mid-points of the compartments from P, are to be found in the Table of Art. 87, alongside the value of n. The values of $1-\cos 2\alpha$ and $\sin 2\alpha$ must be written down and added together; and when substituted in the above

formula applied to the several lunes will give the horizontal attractions for each of them in turn. These added together give the final attraction for the position of P chosen at the outset.

The result of the calculation is as follows:

The horizontal attraction of a slender hemi-spheroidal meniscus of matter at the earth's surface on points 90°, 120°, 135°, 150°, 180° from the pole of the meniscus is

$$0.1202\,\frac{h}{a}g,\quad 0.0412\,\frac{h}{a}g,\quad 0.0236\,\frac{h}{a}g,\quad 0.0138\,\frac{h}{a},\quad 0\,;$$

h = greatest thickness of the meniscus, a = radius of the earth, and the density of the meniscus being half the mean density of the earth. If θ be the distance of any point in the further hemisphere from the pole of the meniscus the above quantities, by the use of indeterminate coefficients, lead to the following formula. Horizontal attraction

$$= (0.1446 \sin\theta + 0.0958 \sin 2\theta + 0.0244 \sin 3\theta)\,\frac{h}{a}g \ldots (1),$$

which may be taken as representing generally the attraction at any point of the hemisphere opposite the meniscus.

By means of Art. 14 it may be shown, that the tangential attraction of the difference of two spheroids of different small ellipticity having the same equator, i.e. of a meniscus in the other hemisphere taken together with the meniscus we have been considering,

$$= \frac{4\pi}{3}\rho a \times \frac{6}{5}\delta\epsilon \cos\phi \sin\phi$$

$$= 0.3 \sin 2\phi\,\frac{h}{a}g,$$

and acts towards the nearest pole; from which also ϕ is measured. Hence if we take the difference of this and (1) we have the attraction of a thin hemi-spheroidal meniscus on a point on its own surface: the formula becomes, attending to the directions of the attraction, Horizontal attraction

$$= (0.1446 \sin\phi + 0.2042 \sin 2\phi + 0.0244 \sin 3\phi)\,\frac{h}{a}g \ldots (2),$$

$\phi = 180^\circ - \theta$, so that θ and ϕ in (1) and (2) are each measured from the pole of the attracting meniscus, and in each case the attraction is reckoned positive towards the pole of the attracting meniscus.

91. A somewhat simpler example for the reader to work out is this, To find the tangential attraction of a hemi-spherical shell of small uniform thickness upon any point in the surface of the whole sphere.

If the calculation be first made for points, as in the last example*, 91°, 120°, 135°, 150°, 180° from the pole of the shell, the results will be

$$1.3750\,\frac{h}{a}g,\quad 0.1916\,\frac{h}{a}g,\quad 0.1128\,\frac{h}{a}g,\quad 0.0852\,\frac{h}{a}g,\quad 0;$$

and the following formula will approximately embrace other points in the hemisphere opposite to the hemi-spherical shell: Horizontal attraction

$$= (2.2054 \sin\theta + 1.9842 \sin 2\theta + 0.7606 \sin 3\theta)\,\frac{h}{a}g \ldots (3).$$

As the tangential attraction of a whole spherical shell on any point is zero, it follows that the tangential attraction of a hemi-spherical shell on any point on its own surface will equal the above with its sign changed: or if ϕ be the angle from the pole of the shell it will be

$$(2.2054 \sin\phi - 1.9842 \sin 2\phi + 0.7606 \sin 3\phi)\,\frac{h}{a}g \ldots (4),$$

θ and ϕ being reckoned in each case from the pole of the attracting shell. In each case the attraction is reckoned positive towards the pole of the attracting meniscus.

92. Suppose we take a hemi-spheroidal meniscus of thickness h at its edge, and no thickness at the pole. The

* The first point is here taken 91° and not 90° (that is, 1° or about 70 miles from the edge of the shell), because otherwise the square of the ratio of the height of the mass on the nearest compartments to the distance from the point attracted could not be neglected. See end of Art. 86.

attraction of this will be found by subtracting the results of Art. 90 from those of Art. 91, they give

$$(2.0608 \sin\theta + 1.8884 \sin 2\theta + 0.7362 \sin 3\theta) \frac{h}{a} g \dots\dots (5),$$

and $$(2.0608 \sin\phi - 2.1884 \sin 2\phi + 0.7362 \sin 3\phi) \frac{h}{a} g \dots (6).$$

93. It has been shown (Art. 89) that the horizontal attraction caused by the Himalaya Mountains is comparatively great. It is possible that, the superabundant matter in mountain-regions having been heaved up from below, or at any rate, having been left aloft as the earth contracted its volume, there may be a deficiency of matter below the mountains which would under certain circumstances have the tendency of counteracting their effect on the plumb-line. This Mr Airy has suggested in a Paper in the *Philosophical Transactions* of 1855, on the hypothesis that the deficiency is immediately below the mountains close to their mass. Upon the supposition that the mountains may have drawn their mass from the regions below through a considerable depth, by an extensive and small expansion of the matter in those lower regions, or, by those regions not having contracted so much as the neighbouring parts have done, the author has calculated the modifying effect on the plumb-line in the *Philosophical Transactions* for 1858—9. This has brought to light the fact, that a trifling deviation in the density from that required for fluid-equilibrium, if it prevail through extensive tracts, may have a sensible effect upon the plumb-line. This has been recently verified, as already noticed (Art. 63), by the observations and calculations of Professor Schmeizer, who has shown that within a distance of 16 miles the plumb-line varies by 16″ near Moscow without any apparent cause, and that it varies in such a way as to indicate a deficiency of matter below. See *Monthly Notices of Ast. Soc.* Ap. 1862. The following Proposition shows, in a general way, how slight such an excess or deficiency of matter need be, if very extensive, to produce a sensible effect on the plumb-line. It also furnishes another application of the method developed in Art. 84. These questions, in themselves interesting as pro-

blems in Attraction, become still more so, as we shall see, in the determination of the Figure of the Earth.

PROP. *To find the effect on the plumb-line of a slight but wide-spread deviation in density in the interior of the earth, either in excess or defect, from that required by the laws of fluid-equilibrium.*

94. Suppose a four-sided space drawn upon the surface of the earth, bounded on two sides by great circles passing through the station where the plumb-line is and making an angle β with each other, the other two sides being parts of circles of which the station is the centre; let ϕ be the angular distance of these two circles measured along the surface, and θ the distance of the middle of ϕ from the station. We shall take $\phi = 2^\circ 52' 40''$ ($=200$ miles), and $\beta = 30^\circ$, and shall find how small θ may be that a mass of small uniform height covering the space should attract the station as if it were collected into the middle point of ϕ. The area of the space

$$= a^2\beta\left\{\cos\left(\theta - \tfrac{1}{2}\phi\right) - \cos\left(\theta + \tfrac{1}{2}\phi\right)\right\} = 2a^2\beta \sin\tfrac{1}{2}\phi \sin\theta,$$

and the chord between the mid-point and the station being $2a \sin\frac{1}{2}\theta$, the attraction of the mass collected at the mid-point and resolved along the tangent

$$= \frac{2\rho h a^2 \beta \sin\frac{1}{2}\phi \sin\theta}{4a^2 \sin^2\frac{1}{2}\theta} \cos\tfrac{1}{2}\theta = \rho h\beta \sin\tfrac{1}{2}\phi \frac{\cos^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}.$$

But by Art. 84 the attraction of the mass

$$= 2\rho h \sin\tfrac{1}{2}\beta\left(\log_e \frac{\sin\frac{1}{2}\theta + \sin\frac{1}{4}\phi}{\sin\frac{1}{2}\theta - \sin\frac{1}{4}\phi} - 2\sin\tfrac{1}{2}\theta \sin\tfrac{1}{4}\phi\right)$$

$$= 4\rho h \sin\tfrac{1}{2}\beta \sin\tfrac{1}{2}\theta \sin\tfrac{1}{4}\phi\left(\frac{1}{\sin^2\frac{1}{2}\theta} + \frac{\sin^2\frac{1}{4}\phi}{3\sin^4\frac{1}{2}\theta} + \dots - 1\right)$$

$$= \rho h\beta \sin\tfrac{1}{2}\phi\left(1 - \frac{\beta^2}{24} + \frac{\phi^2}{32} + \frac{4\sin^2\frac{1}{4}\phi}{3\sin^2\theta}\right)\frac{\cos^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}.$$

This coincides with the previous expression if

$$\frac{4\sin^2\frac{1}{4}\phi}{3\sin^2\theta} - \frac{\beta^2}{24} + \frac{\phi^2}{32} = \text{a very small quantity} = \frac{1}{100} \text{ suppose.}$$

Put $\beta = 30^\circ = \pi \div 6, \quad \phi = 2^\circ\, 52' 40'' = 0.016\pi$;

$$\therefore\ 4 \sin^2 \tfrac{1}{4}\phi = 3 \sin^2 \theta\,(0.010 + 0.012) = 0.066 \sin^2 \theta;$$

$$\therefore\ \sin\theta = 8 \sin \tfrac{1}{4}\phi \text{ nearly}; \quad \therefore\ \theta = 2\phi = 400 \text{ miles}.$$

Hence the centre of the space may be as near as 400 miles to the station, and yet the whole mass be supposed to be collected into its centre. The area $= a^2\beta\phi \sin\theta = 2a^2\beta\phi^2 = (a\phi)^2$ very nearly $= (200)^2$ miles, or the space is equal to a square of 200 miles each way.

95. Now suppose the height of the matter on this space to be 1 mile, and suppose every small vertical prism of it to be distributed uniformly downwards into a slender prism to a depth d. Thus the whole superficial mass 1 mile thick will be distributed through a depth d, and form an attenuated mass the density of which is one d^{th} part of that of the superficial rock. As the mass at the surface may be collected into its middle point, much more may that in any horizontal section of this attenuated mass, because the section is further from the station than the space at the surface. Hence the whole attenuated mass will attract the station as if it were collected uniformly into one vertical prism drawn down from the central point of the surface to the depth d. Let u and v be the distances of the extremities of this prism from the station:—

Therefore attraction on the station along u

$$= \frac{\text{mass}}{uv} = \frac{\rho \times 200^2}{uv}\ (\text{Art. } 55) = \frac{1\,.\,2}{uv}\, g.$$

This will also be approximately the *horizontal* attraction for all distances not exceeding 30° from the station.

Hence deflection of the plumb-line

$$= \frac{1\,.\,2}{uv} \text{ in arc} = \frac{1000000''}{4uv}.$$

96. We may give any values to u and v so long as u is not less than 400 miles. We shall take $u = 400, 600, 800, 1000$ miles successively. The calculation will be facilitated

by using a table of tangents and secants, observing that $u \div d$ is sensibly the tangent of the angle of which $v \div d$ is the secant. Hence the following Table: in constructing which we write down the value of $u \div d$ obtained by mere division, and then find the value of $v \div d$ by finding the secant which corresponds with the tangent indicated by the value of $u \div d$ just written down.

Depth in miles, or d.	Distance of the mid-point of the space from the station, measured along the chord, in miles; viz.							
	$u=400$		600		800		1000	
	$\frac{u}{d}$,	$\frac{v}{d}$	$\frac{u}{d}$,	$\frac{v}{d}$	$\frac{u}{d}$,	$\frac{v}{d}$	$\frac{u}{d}$,	$\frac{v}{d}$
100	4.00	4.12	6.00	6.08	8.00	8.06	10.00	10.05
200	2.00	2.24	3.00	3.16	4.00	4.12	5.00	5.10
300	1.33	1.66	2.00	2.24	2.67	2.85	3.33	3.48
900	0.44	1.09	0.67	1.20	0.89	1.34	1.11	1.49
1000	0.40	1.08	0.60	1.17	0.80	1.28	1.00	1.41

This Table enables us, with the formula above, to tabulate the deflections as follows:

	Distance of mid-point from the station, along the chord, in miles.			
	400	600	800	1000
DEFLECTIONS caused by the mass distributed downwards through a depth of 100 miles.	1″.51	0″.69	0″.39	0″.25
Ditto 200 „	1 .40	0 .66	0 .38	0 .25
Ditto 300 „	1 .25	0 .62	0 .36	0 .24
Ditto 900 „	0 .64	0 .38	0 .26	0 .18
Ditto 1000 „	0 .58	0 .36	0 .24	0 .18

The densities of the masses distributed through the depths 100, 200, 300, 900, 1000 miles are severally inversely proportional to those numbers. Hence by multiplying the lines of

numbers in this table successively by 1, 2, 3, 9, 10 we shall have the deflections of masses having the same volumes as before, but all of the same density, viz. 1-100th part of that of superficial rock. The numbers then are

$1''.51$	$0''.69$	$0''.39$	$0''.25$
2.80	1.32	0.76	0.50
3.75	1.86	1.08	0.72
5.76	3.42	2.34	1.62
5.80	3.60	2.40	1.80

Subtract each line from the line below (except the 3rd line) and we obtain the following:

DEFLECTIONS caused by a semi-cubic mass, 200 miles in each horizontal side and 100 miles deep, density = 1-100th of the density of the surface, and depth of the centre	Distance of the mid-point from the station, along the chord, in miles. 400	600	800	1000
= 50 miles	$1''.51$	$0''.69$	$0''.39$	$0''.25$
Ditto - - 150 „	1 .29	0 .63	0 .37	0 .25
Ditto - - 250 „	0 .95	0 .54	0 .32	0 .22
Ditto - - 950 „	0 .04	0 .18	0 .06	0 .18

The horizontal dimensions of the spaces will be somewhat contracted in passing downwards owing to the convergence of the sides towards the centre of the earth: but the densities from the distribution downwards in slender prisms of uniform mass will increase in a corresponding degree: and the masses of the spaces will be all the same.

The last change we shall make is this. We shall increase the density of the semi-cubic space as its depth increases, so as to make it 1-100th part, not of the superficial density as at present, but of the density of the earth's mass at the centre of the space.

If D be the density of the surface, a the earth's radius, the usually received law of density of the interior is

$$\text{density at depth } d = \frac{2aD}{a-d} \sin\left(\frac{5\pi}{6}\,\frac{a-d}{a}\right),$$

when $d = 50, 150, 250, 950$ miles, this gives the ratio of the density at these depths to the superficial density $= 1.17, 1.21, 1.35, 2.39$. Multiply the deflections last found by these numbers, and we have finally

Deflections caused by an excess or defect of matter prevailing through a semi-cubic space 200 miles in each horizontal side and 100 miles deep, the density of the excess or defect being 1-100th of the earth's density at the centre of the semi-cubic space, when that centre is	Distance of the mid-point of the semi-cubic space from the station, measured along the chord, in miles.			
	400	600	800	1000
50 miles deep	1″.77	0″.81	0″.46	0″.29
Ditto - - 150 ,,	1 .56	0 .76	0 .45	0 .30
Ditto - - 250 ,,	1 .28	0 .73	0 .43	0 .30
Ditto - - 950 ,,	0 .10	0 .43	0 .14	0 .43

The defect or excess in density which we have taken, viz. 1-100th, might have been chosen larger, and the deflections proportionably increased. For there are many kinds of rock, as granite, which differ so in density in the different specimens that the difference between the extremes is greater even than 1-10th of the mean. And if this difference exists at the surface, it does not seem to be improper to suppose that great variations may exist also below, from the effect of the cooling down and solidifying of the crust, even much greater than 1-100th.

97. We have taken a semi-cubic space as our example: but the same result is true of a space of the same volume and of any form so long as its dimensions in one direction are not much larger than in another. This follows from Art. 94. For if the superficial section may be collected into its centre much more may all the other horizontal sections, and the whole horizontal attraction will be nearly equivalent to that of a vertical bar, which may be taken to be the mean diameter of the body.

98. In this Chapter various methods have been given for correcting for the effect of Local Attraction in carrying on Trigonometrical Surveys. If the earth were a perfect spheroid

in form, and either homogeneous in density or lying in spheroidal strata following the law they would assume if the whole were a heterogeneous fluid mass, then gravity would in every place be perpendicular to the spheroidal surface, and follow a fixed definite law as we passed from one latitude to another. But in point of fact the earth does not possess this exact regularity, either in form or density. In every place, therefore, gravity differs slightly, both in intensity and also in direction, from this theoretical value. The difference, either in excess or defect, in each place, is called the Local Attraction at that place.

Much more attention has been paid to this subject of late years than formerly. We will here narrate the circumstances which have led to the problems given on this subject in the present treatise.

The late Major-General Sir George Everest, Superintendent of the Trigonometrical Survey in India, fixed upon Kaliana in the longitude of Cape Comorin, and less than 60 miles from the base of the Himalaya Mountains, as the northern extremity of the Great Indian Arc of Meridian. This he did in the hope that it would be sufficiently far from the mountains to avoid any material effect on the plumb-line arising from their attraction. When, however, he published his final volume in 1847 he pointed out, that the astronomical amplitudes of the first and second divisions of his arc (stretching over about $5^\circ\ 24'$ and $6^\circ\ 4'$ of latitude) were, the one less by $5''.24$, and the other greater by $3''.79$, than the amplitudes measured by the survey operations. His successor asked the author of these pages to turn his attention to this discrepancy, with the view of finding out some explanation of it. The result has been a series of papers in the *Transactions and Proceedings of the Royal Society*, and in the *Philosophical Magazine*, the chief points of which have been reproduced in the present treatise. In the *Transactions* for 1855 he published the method given in Art. 84—87, for finding the horizontal attraction of a given irregular superficial mass, and by applying it to the Himalayas showed that the meridian attraction of the mountain mass, so far from being immaterial, must produce a meridian deflection at the northern extremity of the arc (Kaliana) as large as $28''$, and that the effect must be felt

all the way down to Cape Comorin, and in the centre of India (at Damargida) must produce very nearly 7″, a deflection larger than one which Sir George Everest had detected at Takal Khera, and which induced him to abandon that place as a principal station. Mr Airy, on learning that the effect of the Himalayas is so large, suggested that the mass beneath the mountains is less dense than the mass beneath the plains, grounding his theory on a hypothesis which assumes that the solid crust is very thin, and lighter than the fluid on which it was supposed to rest: see *Phil. Trans.* 1855, p. 101. This would of course counteract the large amount of deflection caused by the mountain mass above: but the data assumed are not satisfactory, as no doubt the earth's crust is not thin (see next Chapter, § 3), also the crust would be heavier than the fluid, by contraction. Moreover in the second communication to the Royal Society, the author showed, by a further application of his method, that the vast Ocean south of India, by reason of the deficiency of density of water compared with rock, must also have a very considerable effect upon the plumb-line at the stations of the Great Arc, to which Mr Airy's explanation would not apply. In the next paper the author made a third application of his method, taking up the idea of deficiency of matter below suggested by Mr Airy, and proved (see Art. 94—97), without any assumptions regarding the thickness of the crust, that a slight, though wide-spread deficiency or excess in the density beneath the surface might anywhere produce local attraction as important as any that was caused by either the mountains or the ocean. In the unknown regions below, then, we seem to have an unlimited resource upon which to draw to explain any anomalies of local attraction we may perceive on the surface. This, however, left the question in a hopeless state of uncertainty, since we are in ignorance of the exact condition of the interior of the earth: and no means were apparent of finding even the resultant effect only of all the causes of disturbance. The two problems, to solve which is the great work of extensive trigonometrical surveys conducted on scientific principles—viz. the accurate Mapping of the surface and the determination of the dimensions of the Mean Figure of the Earth—are both affected by the variation in gravity caused by unknown Local Attrac-

tion; and they are imperfectly solved till some means is found of overcoming the difficulty. This has in a great measure been done in the subsequent papers, as will appear in the second part of this treatise on which we now enter. As we have related the progress of the investigation of this important subject thus far, we will anticipate what is to appear in future pages and complete our narrative.

In a fourth paper published in the *Transactions of the Royal Society*, the author demonstrated that local attraction cannot sensibly affect the *relative* position of places laid down on a map by the Survey operations; so that the maps made by the Survey are internally correct, though the position of the whole map on the earth's surface will depend upon the unknown local deflection in latitude and longitude at the place from which the Survey commences its measures. In a fifth paper, published in the *Proceedings of the Royal Society*, No. 64, in 1864, and in two papers in the January and February numbers of the *Philosophical Magazine*, 1867, the author obtained formulæ for the mean axes of the earth's figure in terms of the unknown local deflections of the plumb-line caused by local attraction at the stations of observation; and by assuming that the three long measured arcs—the Anglo-Gallic, the Russian, and the Indian—are parts of the same ellipse, that is, assuming what all investigations of the Figure of the Earth assume, that the mean figure is an oblate spheroid, the amounts of local deflection of the plumb-line at all the stations of those arcs have been found by the principle of least squares, and the mean axes calculated. The difficulties presented by the existence of unknown local attraction have, therefore, been to a considerable extent surmounted.

The result of the last calculation mentioned above is, that the amount of local attraction is at no station large, when compared with the large separate amounts, positive or negative, produced by the mountains and the ocean, and also extensive though slight variations in the density of the crust. This has led the author to adopt the following hypothesis regarding the Constitution of the Earth's Crust, viz. that the varieties we see in mountains and plains and ocean-beds in the earth's surface have arisen from the earth having been once a fluid or semi-fluid mass, and that in solidifying the

mass has contracted unequally; so as to form hollows where the contraction has been greatest, into which water flowed and formed seas and oceans; and to leave high table-lands and mountain regions, where the contraction has been less. This was enunciated in the paper in the *Proceedings* above noticed. Very recently the author has sent another paper, which is printed in the *Royal Society's Transactions* for 1871, in which he tests this hypothesis regarding the constitution of the crust, by the data now furnished by extensive Pendulum Observations lately conducted in India. These matters are all noticed in the following pages, in Arts. 202—209, 161—174, and 192—196.

Postscript to Article 40.

COR. 1. Let F_i be the most general function of μ and ω which satisfies Laplace's Equation of the ith degree. Then by Arts. 35, 29, $M_0 \ldots M_n$ being known functions of μ and i, and accents indicating the same functions of μ',

$$P_i = M_0 M_0' + \ldots + M_n M_n' \cos n(\omega - \omega') + \ldots;$$

$$\therefore \frac{4\pi}{2i+1} F_i = \int_{-1}^{1} \int_0^{2\pi} F_i' P_i d\mu' d\omega' = \ldots\ldots$$

$$+ M_n \left(\cos n\omega \int_{-1}^{1} \int_0^{2\pi} F_i' M_n' \cos n\omega' d\mu' d\omega'\right.$$

$$\left. + \sin n\omega \int_{-1}^{1} \int_0^{2\pi} F_i' M_n' \sin n\omega' d\mu' d\omega'\right) + \ldots;$$

$$\therefore F_i = C_0 M_0 + \ldots + C_n M_n \cos n(\omega - c_n) + \ldots,$$

$C_0 \ldots C_n$, $c_n \ldots$, being unknown constants.

This is the most general solution of Laplace's Equation. It is of the same form as P_i itself; but the coefficients of the terms are not definite numerical quantities, but are arbitrary constants; and the arguments of the cosines need not have the same epoch.

COR. 2. The simplest forms of Laplace's Functions of the ith order are $M_n \cos n\omega$ and $M_n \sin n\omega$, n having any value of the series 0, 1, 2 ... i.

FIGURE OF THE EARTH.

INTRODUCTION.

99. It is easy to show in a general way, that the earth is a more or less spherical mass.

The globular form is seen in the shadow which the earth casts on the moon in eclipses in a variety of positions. The comparison of the distance at which ships at sea lose sight of each other's decks, with the height of the decks from the water, shows all over the world that the sea is of a globular form; and an approximation to the diameter of the globe is thus obtained by simple geometry. The distance of the horizon at sea as seen from cliffs and hills, the height of which is known, leads to the same result. The distance north and south between two places, measured, for instance, by a perambulator, is always found to be nearly in proportion to the difference of latitude; this could not be the case, if the curve of the meridian were not nearly circular.

After it was known that the earth is of a globular form, Newton was the first who demonstrated that it is not a perfect sphere. From theoretical considerations and also from the discovery that a pendulum moves slower at the equator than in higher latitudes, he arrived at the conclusion that its form is that of an oblate spheroid—the form being derived from rotation in a fluid state. This subject we propose now to consider. There are five ways, as will be seen, in which we can determine the form of the earth. We shall consider them in three Chapters. (1) By calculation on the hypothesis that the earth was once in a fluid state: (2) by means of pendulum experiments, the moon's motion, and precession: and (3) by actual measurement by geodetic operations. It will be seen that the fluid theory has borne its part in each of these, at any rate in pointing out the course to be pursued; for it was from that theory that the notion was borrowed, which is used in the second and third methods, viz. that the mean form is a spheroid.

CHAPTER I.

THE FIGURE OF THE EARTH CONSIDERED AS A FLUID MASS.

§ 1. *The Earth considered to be a fluid homogeneous mass.*

As a first approximation we shall inquire whether a homogeneous fluid mass revolving about a fixed axis can be made to maintain a spheroidal form according to the laws of fluid pressure.

PROP. *A homogeneous mass of fluid in the form of a spheroid revolves with a uniform velocity about an axis: required to determine whether the equilibrium of the surface left free is possible.*

100. Let a and b be the semi-axes of the spheroid referred to three axes of rectangular co-ordinates, b being that about which it revolves: also let $b^2 = a^2(1-e^2)$. The forces which act upon the particle (xyz) are the centrifugal force and the attraction of the spheroid parallel to the axes; these latter are given in Art. 12, and are

$$\frac{2\pi\rho}{e^3}\{\sqrt{1-e^2}\sin^{-1}e - e(1-e^2)\}x,$$

$$\frac{2\pi\rho}{e^3}\{\sqrt{1-e^2}\sin^{-1}e - e(1-e^2)\}y,$$

$$\frac{4\pi\rho}{e^3}\{e - \sqrt{1-e^2}\sin^{-1}e\}z.$$

Let these be represented by Ax, By, Cz. Let w be the angular velocity of the rotation, then $w^2\sqrt{x^2+y^2}$ is the cen-

trifugal force of the particle (xyz), and the resolved parts of it parallel to the axes of x, y, z are w^2x, w^2y, 0. Hence X, Y, Z, the forces acting on (xyz) parallel to the axes, are

$$X = -(A - w^2)\,x, \quad Y = -(B - w^2)\,y, \quad Z = -Cz.$$

These make $Xdx + Ydy + Zdz$ a perfect differential, and therefore so far the equilibrium is possible.

The equation of fluid equilibrium gives

$$\begin{aligned}\frac{1}{\rho}\,dp &= Xdx + Ydy + Zdz \\ &= -(A - w^2)\,(xdx + ydy) - Czdz;\end{aligned}$$

$$\therefore \frac{2p}{\rho} = \text{constant} - (A - w^2)\,(x^2 + y^2) - Cz^2.$$

At the surface $p = 0$, and therefore

$$\frac{A - w^2}{C}\,(x^2 + y^2) + z^2 = \text{const.}$$

is the equation to the surface; and this is a spheroid, and therefore the equilibrium is possible, the form of the spheroid being properly assumed. The eccentricity is given by the condition

$$1 - e^2 = \frac{b^2}{a^2} = \frac{A - w^2}{C},$$

$$\text{or } \frac{w^2}{2\pi\rho} = \frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e - 3\,\frac{1-e^2}{e^2} + \frac{2}{e^3}(1-e^2)^{\frac{3}{2}}\sin^{-1} e;$$

$$\text{or } \frac{w^2}{2\pi\rho} + 3\,\frac{1-e^2}{e^2} - \frac{(3 - 2e^2)\sqrt{1-e^2}}{e^3}\sin^{-1} e = 0.$$

Now observation shows that $\frac{1}{289}$ = the ratio of the centrifugal force at the equator to gravity at the equator. Hence

$$\frac{1}{289} = \frac{w^2 a}{\frac{4}{3}\pi\rho a - w^2 a}; \quad \therefore \frac{w^2}{2\pi\rho} = \frac{1}{435}.$$

By expanding in powers of e and neglecting powers higher than the second, because we know that the earth is nearly spherical, we have

$$\sin^{-1} e = e + \frac{1}{2}\frac{e^3}{3} + \frac{1.3}{2.4}\frac{e^5}{5} + \ldots$$

$$\sqrt{1-e^2} = 1 - \frac{1}{2}e^2 - \frac{1.1}{2.4}e^4;$$

$$\therefore \frac{1}{435} = \left(\frac{3}{e^2} - 2\right)\left(1 - \frac{1}{2}e^2 - \frac{1}{8}e^4\right)\left(1 + \frac{1}{6}e^2 + \frac{3}{40}e^4\right) - \frac{3}{e^2} + 3$$

$$= \left(\frac{3}{e^2} - 2\right)\left(1 - \frac{1}{3}e^2 - \frac{2}{15}e^4\right) - \frac{3}{e^2} + 3 = \frac{4e^2}{15};$$

$$\therefore e^2 = \frac{1}{116}.$$

If ϵ be the ellipticity, then

$$\epsilon = \frac{a-b}{a} = 1 - \sqrt{1-e^2} = \frac{1}{2}e^2 = \frac{1}{232}.$$

This result is so much greater than that obtained by other methods, as we shall see, that it decides against our considering the earth's mass to be homogeneous. Indeed even did we not know that the mean density is as much as twice the density of the surface, it would be *à priori* highly improbable that the mass should be homogeneous, since the pressure must increase in passing towards the centre and the matter be in consequence compressed.

101. Another value of e, nearly $= 1$, satisfies the equation. But this does not give the figure of any of the heavenly bodies, since none of them are very elliptical.

Since there are two values of e which satisfy the equation, it might be supposed that the equilibrium of the mass under one of these forms would be unstable, and, upon any derangement taking place, the fluid would pass to the other as a stable form. But Laplace has shown (*Méc. Céles.* Liv. III. § 21) that for a given primitive impulse there is but one form. In fact it is easily seen that for a given value of w, the angular velocity, the vis viva of two equal masses, so different in

their form as to have e small and nearly equal unity, must be very different, and that therefore the mass cannot pass from one form to the other without a new impulse from without being given to its parts.

102. The relation between w and e in Art. 100, shows that as w alters e alters, and vice versâ. By putting $\frac{dw}{de}=0$, we find the greatest value of w which is consistent with equilibrium. This after some long numerical calculations gives

$$e=\frac{17197}{27197}, \text{ and time of rotation} = 0{\cdot}1009 \text{ day}.$$

103. Before proceeding to calculate the ellipticity on the hypothesis of the earth's mass being heterogeneous we will take the following extreme case. The density increases as we pass down towards the centre. Suppose that at the centre it is infinitely greater than elsewhere: that is, suppose the whole force resides in the centre. The case of nature must lie between this hypothesis and that of the earth's being homogeneous.

PROP. *To calculate the ellipticity of a mass of fluid revolving about a fixed axis and attracted by a force residing wholly in the centre of the fluid and varying inversely as the square of the distance.*

104. Let M be the mass of the fluid; the other quantities as before;

$$\therefore\ X=-\frac{Mx}{r^3}+w^2x, \qquad Y=-\frac{My}{r^3}+w^2y, \qquad Z=-\frac{Mz}{r^3}.$$

Then the equation $Xdx+Ydy+Zdz=0$ becomes

$$\frac{M}{r^3}(xdx+ydy+zdz)-w^2(xdx+ydy)=0;$$

$$\therefore\ \frac{M}{r^2}dr-\frac{w^2}{2}d(x^2+y^2)=0;$$

$$\therefore\ \frac{M}{r}+\frac{w^2}{2}(x^2+y^2)=\text{constant}=C,$$

As in Art. 100,

$$\frac{1}{289}=\frac{w^2a}{\frac{M}{a^2}-w^2a}, \quad \therefore \frac{M}{w^2a^3}=290;$$

$$\therefore \frac{1}{r} \text{ or } \frac{1}{\sqrt{x^2+y^2+z^2}}=\frac{C}{M}-\frac{1}{580}\frac{x^2+y^2}{a^3}.$$

By reversing this, squaring, expanding, and neglecting the square of $\frac{1}{580}$, this is seen to be the equation to a spheroid.

When $x=0$ and $y=0$, then $z=b$; when $z=0$, $x^2+y^2=a^2$;

$$\therefore \frac{1}{b}=\frac{C}{M}, \quad \frac{1}{a}=\frac{C}{M}-\frac{1}{580}\frac{1}{a}, \quad \frac{b}{a}=\frac{580}{581};$$

$$\therefore \epsilon=\frac{1}{581}.$$

This value of ϵ is too small (as we might have expected), as $\frac{1}{232}$ is too large, to agree with the form deduced by actual measurement by geodesy.

105. In a paper in the *Philosophical Transactions* 1838, the late Mr Ivory has shown, as Jacobi and Liouville had done before him, that it is possible to find an ellipsoid of three unequal axes such, that if it be a fluid homogeneous mass and revolves with a certain angular velocity around its least axis, it will maintain its equilibrium. The result he comes to is, that the three axes must be related as follows:

$$c, \quad c\sqrt{1+\lambda^2}, \quad c\sqrt{1+\frac{n^2}{\lambda^2}},$$

where n is a numerical quantity between 1 and 1.9414.

It follows from this relation, that it is impossible that the ellipsoid can be nearly spherical. It can, therefore, in no way whatever assist in explaining the figure of the earth. The problem is not one which can occur in nature, and is purely a geometrical one. Mr Todhunter's critique (*Roy. Soc. Proc.* No. 123) does not touch this result.

§ 2. *The Earth considered to be a fluid heterogeneous mass.*

106. From what has gone before it is clear that the earth's mass is not of uniform density throughout. This result indeed we might have anticipated. We shall now enter upon the more general theory of considering the mass to be heterogeneous in its density: and in this section shall develop the method according to which Laplace considered the problem. He assumed that the earth consists of strata nearly spherical, the solid parts following the same law as the fluid parts. In his integrals he assumes that the density follows a continuous law. We shall show in the next Article, that this being the ease it necessarily follows from well-known facts, that the strata throughout must be all nearly spherical, and that it is not an arbitrary assumption, if the density can be expressed by a function of the co-ordinates.

PROP. *To prove that if the earth were a heterogeneous fluid mass it would lie in strata nearly spherical about the earth's centre.*

107. As the mass is fluid it is clear that it will follow a continuous law of density, which may be expressed by a function of the co-ordinates. The truth of the Proposition rests upon these two facts, which are obtained from observation: (1) That the external surface is nearly spherical; (2) That the force of gravity tends nearly towards the earth's centre. Let r, θ, ω be the co-ordinates from the centre of any point of the surface, $(\cos\theta=\mu)$, and let $r=a+\alpha\,.\,u$, where a is the mean radius, u a function of μ and ω, and α a small constant, the square of which may be neglected because the surface is nearly spherical, $r'\theta'\omega'$ co-ordinates to any point in the interior of the mass, $(\cos\theta'=\mu')$, ρ' the density at this point.

Then (Art. 19) the potential of the whole mass at the point on the surface is

$$V=\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{r'}\frac{\rho' r'^2 d\mu' d\omega' dr'}{\sqrt{r^2+r'^2-2rr'p}},$$

where $p = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega - \omega')$. By expansion this becomes

$$V = \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{r'} \rho' \left\{\frac{r'^2}{r} P_0 + \ldots + \frac{r'^{i+2}}{r^{i+1}} P_i + \ldots\right\} d\mu'\, d\omega'\, dr',$$

where $P_0 \ldots P_i \ldots$ are Laplace's Coefficients.

Put $\rho' = R' + \beta \,.\, U'$, where R' is a function of r' only, independent of μ' and ω', and U' is a function of all three $r'\ \mu'\ \omega'$, β a constant. We have to prove that β is a small quantity of the order of α.

Also suppose

$$\int_0^{r'} r'^{i+2} \rho' dr' = \phi(r') + \beta \,.\, \psi(r', \mu', \omega')$$

the limiting value of r' being $a + \alpha \,.\, u'$,

$$= \phi(a + \alpha u') + \beta\psi(a + \alpha u', \mu', \omega'),$$

$$= A + \alpha B(u'_0 + u'_1 + \ldots\ldots + u'_i + \ldots\ldots)$$

$$+ \beta(\psi'_0 + \psi'_1 + \ldots\ldots \psi'_i + \ldots\ldots),$$

these being series of Laplace's Functions. Then remembering their property proved in Art. 29,

$$V = \int_{-1}^{1}\int_{0}^{2\pi} \left\{C\frac{P_0}{r} + \ldots + \frac{P_i}{r^{i+1}}(\alpha B u'_i + \beta\psi'_i) \ldots\right\} d\mu'\, d\omega'$$

$$= \frac{4\pi}{r} C + \ldots + \frac{4\pi}{(2i+1)\, r^{i+1}} (\alpha B u_i + \beta\psi_i) + \ldots,$$

by the property proved in Art. 35: C is a constant.

Now since the force of gravity acts very nearly towards the centre of the earth, the quantities (see Art. 20), $\frac{dV}{d\mu}$ and $\frac{dV}{d\omega}$, which depend upon the parts of gravity at right angles to r, must both be very small. Hence

$$\alpha B \frac{du_i}{d\mu} + \beta \frac{d\psi_i}{d\mu}, \ \alpha B \frac{du_i}{d\omega} + \beta \frac{d\psi_i}{d\mu}$$

must be small: and this must be the case for all values of μ and ω, that is for every spot on the earth's surface. This cannot be the case unless β be small as well as α.

Hence the terms in ρ' which depend upon μ' and ω' are very small. From this it follows that ρ' may be regarded as a function of $r' + \alpha . \nu'$ where ν' is some function of r', μ', ω';

$$\therefore \; r' + \alpha . \nu' = \text{constant}$$

will be the general equation to layers of equal density. This is evidently the equation to a surface nearly spherical around the origin of r'. Hence the mass lies in strata nearly spherical about the earth's centre.

PROP. *To find the equation of equilibrium of a heterogeneous mass of fluid consisting of strata each nearly spherical, and revolving about a fixed axis passing through the centre of gravity with a uniform angular velocity.*

108. Let XYZ be the sums of the resolved parts of all the forces which act upon any particle (xyz) of the fluid, parallel to the axes of co-ordinates, ρ' the density at that point, p the pressure. Then the equation of fluid equilibrium is

$$\frac{dp}{\rho'} = Xdx + Ydy + Zdz.$$

At the surface, and also throughout any internal stratum of equal pressure, and therefore of equal density, as the effect of temperature is not considered in passing from point to point, $dp = 0$.

Hence $$Xdx + Ydy + Zdz = 0$$

is the differential equation to the exterior surface and to the surfaces of all the internal strata; the particular value assigned to the constant after integration determining to which surface the integral belongs.

The following property belongs to all these surfaces. If ds be the element of any curve drawn on the surface through (xyz), and R be the resultant of XYZ; then the equation may be written

$$\frac{X}{R}\frac{dx}{ds} + \frac{Y}{R}\frac{dy}{ds} + \frac{Z}{R}\frac{dz}{ds} = 0.$$

The first side of this is the cosine of the angle between the resultant and the line ds, and as it equals zero it shows that the resultant force is at right angles to any line in the surface, and therefore to the surface itself at the point (xyz).

The equilibrium will be the same if we suppose the rotatory motion not to exist, but apply to each particle a force equal to the centrifugal force caused by the rotation. The forces then acting on the fluid will be the centrifugal force and the mutual attraction of the parts of the fluid. Let V be the potential (Art. 18) for this mass, then

$$-\frac{dV}{dx}, \quad -\frac{dV}{dy}, \quad -\frac{dV}{dz}$$

are the attractions parallel to the three axes tending towards the origin of co-ordinates. Let w be the angular velocity of rotation about the axis of z, taken as the fixed axis; $w^2 . x$ and $w^2 . y$ will be the centrifugal force at the point (xyz). Then

$$\frac{dp}{\rho'} = \left(\frac{dV}{dx} + w^2 x\right) dx + \left(\frac{dV}{dy} + w^2 y\right) dy + \frac{dV}{dz} dz,$$

$$\therefore \text{ constant or } \int \frac{dp}{\rho'} = V + \frac{w^2}{2}(x^2 + y^2)$$

is the equation to the surface and the strata.

Let r be the distance of the point (xyz) from the origin, and θ the angle r makes with the axis of z, and $\cos\theta = \mu$: then $x^2 + y^2 = r^2 \sin^2\theta = (1 - \mu^2) r^2$. Also let m be the ratio of the centrifugal force at the equator to gravity at the equator $\left(\text{or } \frac{1}{289}\right)$; let a' be the mean radius of the stratum through (xyz); a the mean radius of the surface; then, neglecting small quantities of the second order,

$$m = w^2 \mathrm{a} \div \frac{M}{\mathrm{a}^2} = \frac{w^2 \mathrm{a}^3}{M},$$

$$\text{and } M = 4\pi \int_0^{\mathrm{a}} \rho' a'^2 da' = \frac{4}{3}\pi\phi(\mathrm{a}) \text{ suppose},$$

the strata being considered spherical because of the smallness of the numerator in the value of m;

$$\therefore\ m = \frac{3w^2\mathrm{a}^3}{4\pi\phi(\mathrm{a})}, \quad \therefore\ w^2 = \frac{4\pi}{3} m \frac{\phi(\mathrm{a})}{\mathrm{a}^3},$$

and the equation becomes

$$\text{constant or} \int \frac{dp}{\rho'} = V + \frac{2\pi}{3} m \frac{\phi(\mathrm{a})}{\mathrm{a}^3} (1-\mu^2) r^2$$

$$= V + \frac{4\pi}{9} m \frac{\phi(\mathrm{a})}{\mathrm{a}^3} r^2 + \frac{2\pi}{3} m \frac{\phi(\mathrm{a})}{\mathrm{a}^3} \left(\frac{1}{3} - \mu^2\right) r^2,$$

this arrangement being made, because the second and third terms as they now stand are Laplace's Functions of the order 0 and 2. (See Art. 42, Ex. 1.)

By Art. 53, we have

$$V = \frac{4\pi}{r}\int_0^a \rho' \left\{a'^2 + \frac{d}{da'}\left(\frac{a'^4}{3r} Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1) r^i} Y_i' + \ldots\right)\right\} da'$$

$$+ 4\pi \int_a^{\mathrm{a}} \rho' \left\{a' + \frac{d}{da'}\left(\frac{a'r}{3} Y_1' + \ldots + \frac{r^i}{(2i+1) a'^{i-2}} Y_i' + \ldots\right)\right\} da'.$$

In this put

$$r = a(1 + Y_1 + \ldots Y_i + \ldots) \text{ and } \int_0^a \rho' a'^2 da' = \tfrac{1}{3}\phi(a),$$

as before. Then substitute this value of V in the equation to the strata and equate terms of the order i. (See Art. 38.)

The constant parts give

$$\int \frac{dp}{\rho'} = \frac{4\pi}{3}\frac{\phi(a)}{a} + 4\pi \int_a^{\mathrm{a}} \rho' a' da' + \frac{4\pi}{9} m a^2 \frac{\phi(\mathrm{a})}{\mathrm{a}^3} \ \ldots\ldots \ (1)$$

and the terms of the order i give

$$\frac{\phi(a)}{3a} Y_i - \frac{1}{(2i+1) a^{i+1}} \int_0^a \rho' \frac{d}{da'} (a'^{i+3} Y_i') da'$$

$$- \frac{a^i}{2i+1} \int_a^{\mathrm{a}} \rho' \frac{d}{da'} \left(\frac{Y_i'}{a'^{i-2}}\right) da' = 0, \ \ldots\ldots \ (2)$$

except when $i=2$, in which case the second side is

$$\frac{m}{6}\frac{a^2\phi(a)}{a^3}\left(\frac{1}{3}-\mu^2\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By these equations Y_i is to be calculated, and then the form of the stratum of which the mean radius is a is known by the formula

$$r = a\,(1 + Y_1 + Y_2 + \ldots + Y_i + \ldots).$$

PROP. *To prove that $Y_i = 0$, excepting the case of $i = 2$.*

109. Since Y_i and ρ are functions of a, they may be expanded into ascending series of the form

$$Y = Wa^s + \ldots, \quad \rho = D + D'a^n + \ldots,$$

where D is the density at the centre of the earth, and is as well as W and D' independent of a : s, n ..., must not be negative, otherwise Y and ρ would be infinite at the centre.

Now when these and the corresponding series obtained by putting a' for a, are substituted in the equation of the strata in the last Article, and the first side arranged in powers of a, the various coefficients ought to vanish; excepting when $i=2$, because then the second side is not zero. We shall therefore substitute these series, and search for values of W and s which satisfy the condition.

$$\phi(a) = 3\int_0^a \rho' a'^2 da' = Da^3 + \frac{3D'}{n+3}\,a^{n+3} + \ldots$$

After two easy integrations the equation (2) of the strata becomes

$$WD\left(\frac{1}{3}a^{s+2} - \frac{a^{s-i+2}}{2i+1}\,a^i\right) + \ldots = 0.$$

No value of s will cause these terms to vanish. The only apparent case is when $i=1$, for then by putting $s = i - 2$ the part in the brackets vanishes: but in this particular case $s = -1$, and is negative and therefore inadmissible.

Hence the only way of satisfying the condition is by putting $W = 0$; this shows that Y_i has no first term, that is, that it has no term at all and is therefore zero.

PROP. *To prove that the strata are all spheroidal, concentric, and have a common axis.*

110. By the last two Articles it appears that the equation to the surface is

$$r = a\,(1 + Y_2),$$

and the equation for calculating Y_2 is

$$\frac{\phi(a)}{3a}\,Y_2 - \frac{1}{5a^3}\int_0^a \rho' \frac{d}{da'}\,(a'^5\,Y_2')\,da' - \frac{a^2}{5}\int_a^a \rho' \frac{dY_2'}{da'}\,da'$$

$$= \frac{m}{6}\,\frac{a^2\phi(a)}{a^3}\left(\frac{1}{3} - \mu^2\right).$$

Suppose Y_2 (and similarly Y_2') is expanded in a series of powers of $\frac{1}{3} - \mu^2$, with indeterminate coefficients, to be ascertained by the condition that they shall satisfy the above equation. These coefficients will be functions of a only, as it is seen from the right-hand side of the equation that ω does not enter into the value of Y_2. It is clear that Y_2 consists of only one term, that involving the simple power of $\frac{1}{3} - \mu^2$. Let it be $\epsilon\,(\frac{1}{3} - \mu^2)$, ϵ being a small quantity of the order of m. Hence

$$r = a\,\{1 + \epsilon\,(\tfrac{1}{3} - \mu^2)\},\ \mu = \sin\text{ (latitude)} = \sin l$$

$$= a\,(1 - \tfrac{2}{3}\epsilon)\,(1 + \epsilon\cos^2 l),\ \text{since } \epsilon \text{ is small.}$$

This is the equation to a spheroid from the centre, ϵ being the ellipticity. The axis-minor coincides with the axis of revolution of the whole mass. Hence the strata are concentric spheroids, the minor-axes of which coincide with the axis of revolution of the whole mass.

111. Since the strata are all concentric spheroids with a common axis in the axis of rotation, it follows that the centre of the earth's mass coincides with the centre of the volume, and that the axis of rotation is one of the principal axes of the mass. For this is true of each of the spheroidal strata separately, and is therefore true also of the aggregate or the whole mass.

PROP. *To obtain an equation for calculating the ellipticity of the strata.*

112. Substitute $\epsilon(\frac{1}{3}-\mu^2)$ for Y_2 and $\epsilon'(\frac{1}{3}-\mu^2)$ for Y_2' in equation (3) of Art. 108, and we have, after dividing by $\frac{1}{3}-\mu^2$,

$$\frac{\phi(a)}{3a}\epsilon-\frac{1}{5a^3}\int_0^a\rho'\frac{d}{da'}(a'^5\epsilon')\,da'-\frac{a^2}{5}\int_a^{\mathrm{a}}\rho'\frac{d\epsilon'}{da'}da'=\frac{m}{6}\frac{a^2\phi(\mathrm{a})}{\mathrm{a}^3}.$$

Divide both sides by a^2, and differentiate with respect to a; then multiply by a^6, and differentiate again, and divide by the coefficient of $\frac{d^2\epsilon}{da^2}$;

$$\therefore \frac{d^2\epsilon}{da^2}+\frac{6\rho a^2}{\phi(a)}\frac{d\epsilon}{da}-\left\{1-\frac{\rho a^3}{\phi(a)}\right\}\frac{6\epsilon}{a^2}=0.$$

This may be put into another form. Multiply by $\phi(a)$, then

$$\frac{d}{da}\left\{\phi(a)\frac{d\epsilon}{da}\right\}+\frac{d}{da}\{3\rho a^2\epsilon\}=\frac{6}{a^2}\phi(a)\,\epsilon+3a^2\epsilon\frac{d\rho}{da};$$

$$\text{or }\frac{d^2}{da^2}\{\phi(a)\,\epsilon\}=\frac{6}{a^2}\phi(a)\,\epsilon+3a^2\epsilon\frac{d\rho}{da}.$$

113. COR. By putting $a=\mathrm{a}$ in equation (3) of Art. 108,

$$\int_0^{\mathrm{a}}\rho'\frac{d}{da'}(a'^5\epsilon')\,da'=\frac{5}{3}\mathrm{a}^2\phi(\mathrm{a})\left(\epsilon-\frac{m}{2}\right).$$

PROP. *To prove that the ellipticity of the strata decreases from the surface towards the centre.*

114. We assume that the density of the earth increases from the surface to the centre. Let then $\rho=D-Ea^n+\ldots$, where E is positive: and $\epsilon=A+Ba^m+\ldots$. Then

$$\frac{\rho a^3}{\phi(a)}=1-\frac{n}{n+3}\frac{E}{D}a^n+\ldots=1-Ha^n+\ldots,\ H\text{ positive.}$$

Put these in the differential equation in ϵ of Art. 112,

$$B(m^2+5m)\,a^{m-2}-6AHa^{n-2}+\ldots=0.$$

Neither m nor B can equal zero, because then the second term of ϵ only merges into the first. Nor can $m=-5$, a negative quantity. Hence the first term will not vanish of itself. But we may make the first and second vanish together by putting $n=m$ and $B(m^2+5m)=6AH$. Hence B must be positive; i.e. *near the centre* ϵ increases towards the surface. Suppose it attains a *maximum*, and then decreases. At this point $\frac{d\epsilon}{da}=0$; and the equation of Art. 112 gives

$$\frac{d^2\epsilon}{da^2}=\left\{1-\frac{\rho a^3}{\phi(a)}\right\}\frac{6\epsilon}{a^2},\ \text{a } \textit{positive} \text{ quantity.}$$

This corresponds to a *minimum*. Hence ϵ does not attain a maximum, and therefore it continually increases from the centre to the surface. In the above we have assumed that $\phi(a)$ is greater than ρa^3. This appears

$$\because\ \phi(a)=3\int_0^a\rho' a'^2 da=\rho a^3-\int_0^a a'^3\frac{d\rho'}{da'}da',$$

$$\text{and } \frac{d\rho'}{da'} \text{ is negative by hypothesis.}$$

The calculation cannot be carried further before we determine upon the law of density in the distribution of the earth's mass.

PROP. *To find a law of density of the earth's mass.*

115. In order to make the equation in Art. 112 for determining the ellipticity integrable, it is necessary to assume a law of density of the mass of the earth. Experiment has not yet determined what the law of compression in such a mass would be. It is necessary to make some assumption.

In order to make the equation

$$\frac{d^2}{da^2}\{\phi(a)\,\epsilon\}=\frac{6}{a^2}\phi(a)\,\epsilon+3a^2\epsilon\frac{d\rho}{da}$$

integrable, we must assume the last term to be some multiple of $\phi(a)\,\epsilon$, the other factor being a function of a. Suppose

$$3a^2\frac{d\rho}{da} = -q^2\phi(a) = -3q^2\int_0^a \rho' a'^2 da',$$

the negative sign being taken because the density decreases from the centre to the surface; q^2 is some function of a. The equation for the ellipticity becomes

$$\frac{d^2\{\phi(a)\,\epsilon\}}{da^2} + \left(q^2 - \frac{6}{a^2}\right)\phi(a)\,\epsilon = 0 \;\ldots\ldots\ldots\ldots (1).$$

The only case in which this equation has been integrated is when q^2 is constant. We must therefore assume it to be so*. The equation for the density then gives by differentiation and re-adjustment

$$\frac{d^2 . \rho a}{da^2} + q^2 . \rho a = 0;$$

$$\therefore\ \rho a = Q\sin(qa + A).$$

But ρ would be infinite at the centre unless $A = 0$. Hence

$$\rho = Q\,\frac{\sin qa}{a}.$$

In the case of the earth the constants Q and q must be found from the conditions, that the density at the surface is the density of granite; and that the mean density of the whole is twice that density. This last leads to the formula,

$$2 = \text{mean density} \div \text{superficial density}$$

$$= \text{mass} \div (\text{volume} \times \text{superficial density})$$

* In order to explain how to differentiate a definite integral with respect to a quantity involved in the limits, let $\int f(x) = F(x) + \text{const.}$;

$$\therefore\ \int_b^a f(x)\,dx = F(a) - F(b);$$

$$\therefore\ \frac{d}{da}\int_b^a f(x)\,dx = \frac{dF(a)}{da} = f(a);\quad \frac{d}{db}\int_b^a f(x)\,dx = -\frac{dF(b)}{db} = -f(b).$$

$$= 4\pi \int_0^{\text{a}} \rho a^2 da \div \left(\frac{4\pi}{3} \text{a}^3 . Q \frac{\sin q\text{a}}{\text{a}}\right)$$

$$= 3 \int_0^{\text{a}} a \sin qa\, da \div \text{a}^2 \sin q\text{a} = \frac{3}{q^2\text{a}^2}\left(1 - \frac{q\text{a}}{\tan q\text{a}}\right).$$

This equation is satisfied by

$$q\text{a} = 2.4605, \ = 140^\circ\, 58'\, 35''.$$

If D be the density of the surface or 2.75, then

$$Q = \frac{D\text{a}}{\sin q\text{a}} = \frac{2.75\text{a}}{0.63} = 4.365\text{a},$$

and also $Qq = 10.74$.

116. If the law of density deduced in Art. 115 be used in equation (1) of Art. 108, then, neglecting the small term,

$$\int_0^a \frac{1}{\rho'} \frac{dp}{da'} da' = \frac{4\pi}{3} \frac{\phi(a)}{a} + 4\pi \int_a^{\text{a}} \rho' a' da'$$

$$= \frac{4\pi}{a} Q \int_0^a a' \sin qa' da' + 4\pi Q \int_a^{\text{a}} \sin qa' da'$$

$$= \frac{4\pi}{q} Q \left(\frac{\sin qa}{qa} - \cos qa - \cos q\text{a} + \cos qa\right)$$

$$= \frac{4\pi}{q^2} \rho - \frac{4\pi}{q} Q \cos q\text{a}.$$

Hence by differentiation with respect to a,

$$\frac{dp}{da} = \frac{4\pi}{q^2} \rho \frac{d\rho}{da} = \frac{2\pi}{q^2} \frac{d.\rho^2}{da},$$

or the increase of pressure varies as the increase in the square of the density. It was by assuming this law between pressure and density, that Laplace deduced the law of density which we have arrived at in another way, viz. by finding what condition is necessary to make the equation of ellipticities integrable (*Mémoires de l'Institut*, Tom. III. p. 496).

PROP. *To find an expression for the ellipticity of the strata, with the law of density deduced in the last Proposition, and to reduce it to numbers for the surface.*

117. The equation of ellipticities (from Art. 115) is

$$\frac{d^2}{da^2}\{\phi(a)\epsilon\}\,\frac{6}{a^2}+\{q^2-\phi(a)\epsilon\}.$$

In order to integrate this put

$$\phi(a)\epsilon=\frac{1}{a^2}\int_0^a a'\int_0^{a'} a'x'da'^2;$$

$$\therefore\ \frac{d\,.\,\phi(a)\epsilon}{da}=-\frac{2}{a^3}\int_0^a a'\int_0^{a'} a'x'da'^2+\frac{1}{a}\int_0^a a'x'da';$$

$$\therefore\ \frac{d^2.\,\phi(a)\epsilon}{da^2}=\frac{6}{a^4}\int_0^a a'\int_0^{a'} a'x'da'^2-\frac{2}{a^2}\int_0^a a'x'da'-\frac{1}{a^2}\int_0^a a'x'da'+x;$$

$$\therefore\ x-\frac{3}{a^2}\int_0^a a'x'da'+\frac{q^2}{a^2}\int_0^a a'\int_0^{a'} a'x'da'^2=0.$$

Multiply by a^2 and differentiate;

$$\therefore\ a^2\frac{dx}{da}+2ax-3ax+q^2a\int_0^a a'x'da'=0.$$

Divide by a and differentiate, and then divide by a;

$$\frac{d^2x}{da^2}+q^2x=0.$$

The solution of this is

$$x+Cq^2\sin(qa+B)=0,$$

C and B being independent of a;

$$\therefore\ \int_0^a a'x'da'=Cqa\cos(qa+B)-C\sin(qa+B);$$

$$\therefore \ \phi(a)\,\epsilon = \frac{1}{a^2}\left\{Ca^2 \sin(qa+B) + \frac{2C}{q} a \cos(qa+B) \right.$$

$$\left. - \frac{2C}{q^2}\sin(qa+B) + \frac{C}{q} a\cos(qa+B) - \frac{C}{q^2}\sin(qa+B)\right\}$$

$$= C\left\{\left(1-\frac{3}{q^2a^2}\right)\sin(qa+B) + \frac{3}{qa}\cos(qa+B)\right\}.$$

In our case $B=0$, otherwise the ellipticity at the centre would be infinite, as is easily seen by expanding ϵ in powers of a.

$$\text{Since } \phi(a) = 3\int_0^a \rho' a'^2 da' = \frac{3Q}{q^2}(\sin qa - qa\cos a)$$

$$\text{ellipticity} = \frac{Cq^2}{3Q}\,\frac{\left(1-\dfrac{3}{q^2a^2}\right)\tan qa + \dfrac{3}{qa}}{\tan qa - qa}.$$

And the ratio of this to the ellipticity of the surface

$$= \frac{\tan q\mathrm{a} - q\mathrm{a}}{\tan qa - qa}\,\frac{\left(1-\dfrac{3}{q^2a^2}\right)\tan qa + \dfrac{3}{qa}}{\left(1-\dfrac{3}{q^2\mathrm{a}^2}\right)\tan q\mathrm{a} + \dfrac{3}{q\mathrm{a}}}.$$

This gives the law of decrease in the ellipticity of the strata in passing down from the surface to the centre.

By Art. 113, ϵ being now the ellipticity of the surface,

$$\frac{5}{3}\mathrm{a}^2\phi(\mathrm{a})\left(\epsilon - \frac{1}{2}m\right) = \int_0^{\mathrm{a}} \rho' \frac{d}{da'}(a'^5\epsilon')\,da' = Q\int_0^{\mathrm{a}} \frac{\sin qa'}{a'}\frac{d}{da'}(a'^5\epsilon')\,da'$$

$$= Q\left\{\mathrm{a}^4\epsilon\sin q\mathrm{a} + \int_0^{\mathrm{a}} a'^3\epsilon'(\sin qa' - qa'\cos qa')\,da'\right\} \text{ by parts.}$$

Substituting for ϵ' from the ratio of ellipticities above, integrating and reducing, the integral in this expression

$$= \frac{\epsilon}{q^4}\,\frac{(\tan q\mathrm{a} - q\mathrm{a})\sin q\mathrm{a}}{\left(1-\dfrac{3}{q^2\mathrm{a}^2}\right)\tan q\mathrm{a} + \dfrac{3}{q\mathrm{a}}}\left\{6q^2\mathrm{a}^2 - 15 - \frac{q^3\mathrm{a}^3 - 15q\mathrm{a}}{\tan q\mathrm{a}}\right\}.$$

Substituting for ϕ (a), $\frac{m}{2\epsilon} =$

$$1 - \frac{q^4 a^4 - 3q^2 a^2 + \frac{3q^3 a^3}{\tan qa} + \left(1 - \frac{qa}{\tan qa}\right)\left(6q^2 a^2 - 15 - \frac{q^3 a^3 - 15qa}{\tan qa}\right)}{5\left(1 - \frac{qa}{\tan qa}\right)\left(q^2 a^2 - 3 + \frac{3qa}{\tan qa}\right)}$$

$$= \frac{2q^2 a^2 - q^4 a^4 - \frac{q^3 a^3}{\tan qa} - \frac{q^4 a^4}{\tan^2 qa}}{5\left(1 - \frac{qa}{\tan qa}\right)\left(q^2 a^2 - 3 + \frac{3qa}{\tan qa}\right)}.$$

Put $\frac{qa}{\tan qa} = 1 - z$ to facilitate the calculation;

$$\therefore\ \epsilon = \frac{5m}{2} \frac{z(q^2 a^2 - 3z)}{-q^4 a^4 + 3q^2 a^2 z - q^2 a^2 z^2} = \frac{5m}{2} \frac{1 - \frac{3z}{q^2 a^2}}{3 - z - \frac{q^2 a^2}{z}}.$$

When this is calculated for the surface, we shall be able to find the ellipticity of any stratum we like by the ratio of ellipticities found above.

118. To reduce this to numbers put, as before, $qa = 2.4605$,

$$= 140^\circ 58' 35''; \text{ then } \tan qa = -0.8105,\ z = 4.0359,$$

$$q^2 a^2 = 6.0541,\ q^2 a^2 \div z = 1.5.$$

Hence the above formula for ϵ gives

$$\epsilon = \frac{5m}{2} \frac{1}{2.5359} = \frac{m}{1.01436} = \frac{1}{293}, \text{ since } m = \frac{1}{289}.$$

119. Cor. 1. We will find the numerical value of the ellipticity of some of the strata, according to this law of density, with a view of showing at what rate the ellipticity diminishes in descending towards the centre. We will suppose the mass divided into four shells and a central nucleus, the

radius of the nucleus and the thickness of each shell being equal to one-fifth of the earth's radius. Taking the value of qa found above, then for the outer surface and the four other surfaces we have

$$qa = 140°58'35'',\ 112°46'52'',\ 84°35'9'',\ 56°23'26'',\ 28°11'43''$$

$$= 2.4605,\ 1.9684,\ 1.4763,\ 0.9842,\ 0.4921 \text{ in arc; and}$$

$$\frac{qa}{\tan qa} = -3.035906,\ -0.826756,\ 0.139920,\ 0.654135,\ 0.917944,$$

$$z = 4.035906,\ 1.826756,\ 0.860080,\ 0.345865,\ 0.082056,$$

$$\frac{1}{z} = 0.2478,\ 0.5474,\ 1.1627,\ 2.8913,\ 12.1868,$$

$$\frac{3}{q^2a^2} = 0.4955,\ 0.7743,\ 1.3765,\ 3.0971,\ 12.3884.$$

$$\therefore \frac{1}{z} - \frac{3}{q^2a^2} = -0.2477,\ -0.2269,\ -0.2138,\ -0.2058,\ -0.2016.$$

Hence by Art. 117

$$\frac{\text{ellipticity}}{\text{ellipticity of surface}} = 1,\ \frac{2269}{2477},\ \frac{2138}{2477},\ \frac{2058}{2477},\ \frac{2016}{2477}$$

$$= 1,\ \frac{1}{1.09},\ \frac{1}{1.16},\ \frac{1}{1.20},\ \frac{1}{1.23}.$$

120. Cor. 2. In a future Article we shall require to know the masses of the portions of the earth which lie within these surfaces to a first approximation, that is, neglecting their ellipticity. We will therefore calculate them here. Let E be the whole mass of the earth, and M the mass of a portion of it of which a is the mean radius. Then, if $Q \sin qa \div a$ is the density,

$$M = \int_0^a 4\pi a^2 \cdot Q \frac{\sin qa}{a}\, da = 4\pi Q \left(-\frac{a}{q} \cos qa + \frac{1}{q^2} \sin qa \right);$$

$$\therefore \frac{M}{E} = 1,\ 0.6628,\ 0.3371,\ 0.1134,\ 0.0153.$$

Hence the ratios of the masses of the four shells and of the nucleus to the whole mass are

$$0.3372,\ 0.3257,\ 0.2237,\ 0.0981,\ 0.0153.$$

And the ratios of the mean densities of these to the mean density of the whole earth equal these multiplied by the fractions

$$\frac{5^3}{5^3-4^3},\ \frac{5^3}{4^3-3^3},\ \frac{5^3}{3^3-2^3},\ \frac{5^3}{2^3-1},\ \frac{5^3}{1^3};$$

and are, therefore,

$$0.6910,\ 1.1003,\ 1.4717,\ 1.7518,\ 1.9125.$$

PROP. *To find the potential of the earth for an external point, on the hypothesis of the arrangement of the mass being according to the fluid law.*

121. By putting $a = \text{a}$ in the formula for the potential of the earth given in Art. 108, it becomes, for an external point, bearing in mind Art. 109,

$$V = \frac{4\pi}{r}\int_0^{\text{a}} \rho' \left\{a'^2 + \frac{d}{da'}\left(\frac{a'^5}{5r^2}Y_2'\right)\right\} da',$$

$$= \frac{4\pi}{r}\int_0^{\text{a}} \rho' \left\{a'^2 + \frac{1}{5r^2}\frac{d(a'^5\epsilon')}{da'}\left(\frac{1}{3}-\mu^2\right)\right\} da', \text{ by Art. 110.}$$

Also by Art. 113,

$$\int_0^{\text{a}} \rho' \frac{d(a'^5\epsilon')}{da'} da' = \frac{5}{3}\text{a}^2\phi(\text{a})\left(\epsilon - \frac{m}{2}\right) = \frac{5}{4\pi}\text{a}^2 E\left(\epsilon - \frac{m}{2}\right);$$

$$\therefore\ V = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right)\frac{E\text{a}^2}{r^3}\left(\frac{1}{3}-\mu^2\right),$$

for an external point, E being the mass.

PROP. *To find the law of gravity at the surface of a spheroid of equilibrium and of small ellipticity; and to prove the relation between the ellipticity and gravity at the pole and equator, commonly called* Clairaut's Theorem.

122. The potential of the earth for an external point is

$$V=\frac{E}{r}+\left(\epsilon-\frac{m}{2}\right)\frac{Ea^2}{r^3}\left(\frac{1}{3}-\mu^2\right), \text{ (Art. 121).}$$

Let g be gravity. Then since the angle between the radius vector r and the normal varies as the ellipticity and therefore its cosine must be taken $=1$, the value of gravity is $-\frac{dV}{dr}$ $-$ the part of the centrifugal force resolved along r

$$=\frac{E}{r^2}+3\left(\epsilon-\frac{m}{2}\right)\frac{Ea^2}{r^4}\left(\frac{1}{3}-\mu^2\right)-w^2r(1-\mu^2)$$

$$=\frac{E}{r^2}+3\left(\epsilon-\frac{m}{2}\right)\frac{Ea^2}{r^4}\left(\frac{1}{3}-\mu^2\right)-m\frac{E}{a^3}r\left(\frac{2}{3}+\frac{1}{3}-\mu^2\right).$$

Substitute for r and omit small quantities of the second order,

$$\therefore\ g=\frac{E}{a^2}\left(1-\frac{2}{3}m\right)-\frac{E}{a^2}\left(\frac{5}{2}m-\epsilon\right)\left(\frac{1}{3}-\mu^2\right)$$

$$=\frac{E}{a^2}\left\{1+\frac{1}{3}\epsilon-\frac{3}{2}m+\left(\frac{5}{2}m-\epsilon\right)\sin^2.\text{latitude}\right\}$$

$$=G\left\{1+\left(\frac{5}{2}m-\epsilon\right)\sin^2 l\right\},$$

where G is gravity at the equator.

Hence the increase of gravity in passing from the equator to the poles varies as the square of the sine of the latitude.

This expression immediately leads to the following property, called *Clairaut's Theorem*, after its discoverer.

$$\frac{\text{Polar gravity}-\text{equatorial gravity}}{\text{equatorial gravity}}+\text{ellipticity}$$

$$=\left(\frac{5}{2}m-\epsilon\right)+\epsilon=\frac{5}{2}m$$

$$=\frac{5}{2}\times\text{ratio of centrifugal force at equator to gravity.}$$

123. The expression for g obtained above, when compared with the value of gravity obtained from pendulum experi-

ments, furnishes a test of the correctness of the results to which the fluid theory leads. It will appear in the end, however, that the converse of this is not necessarily true, viz. that if the results are true, that is, if the increase of gravity along the surface varies as the square of the sine of the latitude, and if Clairaut's Theorem is found to be true on the earth's surface, the fluid theory is true. For in the next Chapter we shall show that the above formula for g can be obtained by a process independent of the fluid theory. We shall, therefore, defer till we come to that Chapter the consideration of the evidence which the pendulum furnishes on the subject of the earth's figure; only adding here, for another purpose, a Table giving an extract of the results of pendulum experiments made by Major General Sabine many years ago, and published in a work entitled, *Account of Experiments to determine the Figure of the Earth by means of the Pendulum Vibrating Seconds in different Latitudes*, 1825. This abstract

VIBRATIONS.

Stations.	Latitude.	Vibrations. Computed.	Vibrations. Observed.	Differences.
		s.	s.	s.
Equator	0° 0′ 0″	86263,60		
St Thomas	0 24 41 N.	86263,60	86269,32	+5,72
Maranham	2 31 34 S.	86264,30	86259,77	−4,53
Ascension	7 55 30 S.	86267,86	86273,04	+5,18
Sierra Leone	8 29 28 N.	86268,48	86268,33	−0,15
Trinidad	10 38 55 N.	86271,24	86267,27	−3,97
Bahia	12 59 21 S.	86274,90	86273,16	−1,74
Jamaica	17 56 7 N.	86284,80	86285,12	+0,32
New York	40 42 43 N.	86358,66	86357,73	−0,93
Paris	48 50 14 N.	86390,20	86388,48	−1,72
Shanklin	50 37 24 N.	86397,06	86396,54	−0,52
Greenwich	51 28 40 N.	86400,34	86400,59	+0,25
London	51 31 8 N.	86400,48	86400,00	−0,48
Arbury	52 12 55 N.	86403,12	86403,31	+0,19
Clifton	53 27 43 N.	86407,80	86407,23	−0,57
Altona	35 32 45 N.	86408,10	86408,94	+0,84
Leith	55 58 41 N.	86417,02	86417,89	+0,87
Portsoy	57 40 59 N.	86423,10	86424,60	+1,50
Unst	60 45 28 N.	86433,64	86435,56	+1,92
Drontheim	63 25 54 N.	86442,24	86438,77	−3,47
Hammerfest	70 40 5 N.	86462,42	86461,05	−1,37
Greenland	74 32 19 N.	86471,00	86470,50	−0,50
Spitzbergen	79 49 54 N.	86479,90	86483,01	+3,11

is taken from his translation of the *Cosmos*, Vol. IV. Part I. The column of "computed" vibrations assumes that the change of gravity varies as the change in the square of the sine of the latitude, and the last column shows how far the experiments confirm this.

The ratio of increase of gravity from the equator to the pole deduced from the observations is 0.0051828, (see *Cosmos*, p. 468).

124. We will make use of these data to ascertain what effect certain hypothetical redistributions of the mass, differing from the fluid arrangement, would have upon the motion of the pendulum. The results furnish a kind of test of the sensitiveness of the pendulum as an instrument for detecting the effect of departures from the fluid arrangement. Of course if these departures happened to coincide with any of those peculiar laws of redistribution of the mass which would leave the external attraction the same as before, the pendulum would not detect them.

PROP. *To find the effect on the pendulum of certain hypothetical changes in the distribution of the materials of the earth's mass.*

125. We will suppose the earth's mass divided into four shells and a nucleus, the radius of the nucleus and the thickness of each shell being equal to one-fifth of the earth's radius, or about 800 miles. We shall make three separate hypotheses, and examine the effect of each:—

(1) That the masses of the second and third shells are both altered, each in a different proportion, so as to preserve the whole mass the same and not to alter the form of the strata.

(2) That the form of the strata in one of the shells is altered without affecting the mass.

(3) That the earth consists of a homogeneous mass of the same density as the surface, with the remainder of the mass distributed according to any law in spherical shells.

126. *First re-arrangement.* Let $EE_1E_2E_3E_4$ be the masses of the earth and of the portions of it lying within the inner surfaces of the four successive shells: $VV_1V_2V_3V_4$ the corresponding potentials for a point at distance r from the centre and in latitude of which the sine is μ. Then, by Art. 121,

$$V = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right)\frac{Ea^2}{r^3}\left(\frac{1}{3} - \mu^2\right),$$

and $V_1V_2\ldots$ have corresponding values, $\epsilon_1\epsilon_2\ldots$ $m_1m_2\ldots$ being similar quantities to ϵ and m. Then $V_1 - V_2$ and $V_2 - V_3$ are the potentials of the second and third shells. Also $E_1 - E_2$, $E_2 - E_3$ are the masses of those shells. Suppose the first of these masses is altered in the ratio $\alpha : 1$, and the second in the ratio $\beta : 1$; then as the total mass is unaltered, by hypothesis,

$$\alpha(E_1 - E_2) + \beta(E_2 - E_3) = E_1 - E_3,$$

$$\text{or } 1 - \beta = (\alpha - 1)\frac{E_1 - E_2}{E_2 - E_3}.$$

In consequence of this change the potentials of the shells become $\alpha(V_1 - V_2)$ and $\beta(V_2 - V_3)$. Hence if U be the potential of the whole earth thus altered in the arrangement of its materials

$$U = V + (\alpha - 1)(V_1 - V_2) + (\beta - 1)(V_2 - V_3)$$

$$= \frac{E}{r} + \frac{Ea^2}{r^3}\left[\epsilon - \frac{m}{2} + (\alpha - 1)\left\{\frac{E_1}{E}\frac{a_1^2}{a^2}\left(\epsilon_1 - \frac{m_1}{2}\right) - \frac{E_2}{E}\frac{a_2^2}{a^2}\left(\epsilon_2 - \frac{m_2}{2}\right)\right\}\right.$$

$$\left. + (\beta - 1)\left\{\frac{E_2}{E}\frac{a_2^2}{a^2}\left(\epsilon_2 - \frac{m_2}{2}\right) - \frac{E_3}{E}\frac{a_3^2}{a^2}\left(\epsilon_3 - \frac{m_3}{2}\right)\right\}\right]\left(\frac{1}{3} - \mu^2\right).$$

As m is the ratio of the centrifugal force to gravity at the equator of the spheroid,

$$\therefore \frac{m_1}{m} = \frac{E}{E_1}\frac{a_1^3}{a^3}, \quad \frac{m_2}{m} = \frac{E}{E_2}\frac{a_2^3}{a^3}, \quad \frac{m_3}{m} = \frac{E}{E_3}\frac{a_3^3}{a^3};$$

$$\therefore U = \frac{E}{r} + \frac{Ea^2}{r^3}\left\{\epsilon - \frac{m}{2} + L\right\}\left(\frac{1}{3} - \mu^2\right),$$

where $L=$

$$(\alpha-1)\left\{\frac{E_1}{E}\frac{a_1^2}{\mathrm{a}^2}\epsilon_1-\frac{E_2}{E}\frac{a_2^2}{\mathrm{a}^2}\epsilon_2\frac{E_1-E_3}{E_2-E_3}+\frac{E_3}{E}\frac{a_3^2}{\mathrm{a}^2}\epsilon_3\frac{E_1-E_2}{E_2-E_3}\right.$$

$$\left.-\frac{m}{2}\left(\frac{a_1^5}{\mathrm{a}^5}-\frac{a_2^5}{\mathrm{a}^5}\frac{E_1-E_3}{E_2-E_3}+\frac{a_3^5}{\mathrm{a}^5}\frac{E_1-E_2}{E_2-E_3}\right)\right\}.$$

By substituting the values found in Art. 119, 120, according to the fluid-hypothesis, we have

$$L=(\alpha-1)\,\epsilon\left(\frac{0.6628}{1.09}\frac{16}{25}-\frac{0.3371}{1.16}\frac{9}{25}\frac{5494}{2237}+\frac{0.1134}{1.20}\frac{4}{25}\frac{3257}{2237}\right)$$

$$-(\alpha-1)\frac{m}{2}\left(\frac{1024}{3125}-\frac{243}{3125}\frac{5494}{2237}+\frac{32}{3125}\frac{3257}{2237}\right)$$

$$=(\alpha-1)\left(\frac{0.3892-0.2569+0.0220}{294}\right.$$

$$\left.-\frac{0.3277-0.1910+0.0149}{578}\right)$$

$$=(\alpha-1)\,(0.0005316-0.0002623)=(\alpha-1)\times 0.0002693,$$

the value of ϵ here used being taken from the British Ordnance Survey. See also Art. 118.

Now gravity $=-\dfrac{dV}{dr}-$ centrifugal force, at the surface.

Hence the ratio of gravity, as altered by this change, to gravity as it is

$$=\left(-\frac{dU}{dr}-\text{centrifugal force}\right)\div\left(-\frac{dV}{dr}-\text{centrifugal force}\right)$$

$$=1+3L\left(\frac{1}{3}-\mu^2\right),$$

and the increase in passing from the equator to the pole

$$=3L=(\alpha-1)\times 0.0008079$$

$$=(\alpha-1)\frac{0.0008079}{0.0051828}\times \text{actual increase (see Art. 123),}$$

$$=\frac{\alpha-1}{6.4}\times \text{actual increase.}$$

The table in Art. 123 shows that between the equator and Spitzbergen in about 80° north latitude (the highest place north where pendulum experiments have been made) 214 vibrations are lost in 24 hours by a seconds' pendulum. Hence the number which would be lost from the re-arrangement of the mass now under consideration would equal

$$\frac{\alpha-1}{6\cdot 4}\times 214=33\,(\alpha-1).$$

We may suppose that a difference of 5 beats of the pendulum at the equator and at Spitzbergen would hardly be detected by this means, as that is about the greatest difference in the Table between the observed and computed vibrations. Put then

$$33\,(\alpha-1)=5,\ \text{ or } \alpha=1+\frac{1}{7},\ \text{ and } \beta=1-\frac{1}{7}\times\frac{3257}{2237}=1-\frac{1}{5}$$

nearly, that is, the density of the second may be increased by as much as 1-7th and that of the third be diminished by as much as 1-5th, without our detecting the deranging effect on the pendulum. This is intelligible; for the ellipticities being all very small, the change of densities proposed in this first hypothesis is very much the same as mere spherical concentration or dispersion, which we know (Art. 3) may be carried on to any extent without producing any external effect.

127. *Second re-arrangement.* Suppose that all the strata in one of the shells lose their ellipticity and become spherical, the parts about the poles of the upper surface of the shell swelling up and penetrating the mass of the shell above it, and the parts about the equator of the lower surface of the shell penetrating the shell within; so that the original sphe-

roidal shell may become a spherical shell of the same mass as before, and its mass still co-existing with the other shells and nucleus. This change amounts simply to this: the density of the mass is doubled through a thin space of the form of two hemispherical meniscuses, the rims of which are of no thickness and touch each other at the equator of the upper surface of the shell in question, the thickness of the meniscuses being greatest at the poles and equalling the compression of that surface; and the density is also doubled through a space at the lower surface of the shell generated by the revolution of a crescent round the earth's axis of which the width at the middle equals the distance of the equator of that lower surface from the inscribed sphere. The matter causing the doubling of the density through these thin spaces is drawn from the original shell itself.

We will take the second shell for our example, and will apply the result to the other shells.

The potential of this shell is $V_1 - V_2$; this must now be replaced by $(E_1 - E_2) \div r$, the potential of a shell of spherical strata. Hence

$$U = V - (V_1 - V_2) + (E_1 - E_2) \div r$$

$$= \frac{E}{r} + \frac{E\mathrm{a}^2}{r^3}\left\{\epsilon - \frac{m}{2} - \left(\epsilon_1 - \frac{m_1}{2}\right)\frac{E_1}{E}\frac{a_1^2}{\mathrm{a}^2} + \left(\epsilon_2 - \frac{m_2}{2}\right)\frac{E_2}{E}\frac{a_2^2}{\mathrm{a}^2}\right\}\left(\frac{1}{3} - \mu^2\right)$$

$$= \frac{E}{r} + \frac{E\mathrm{a}^2}{r^3}\left\{\epsilon - \frac{m}{2} - \frac{E_1}{E}\frac{a_1^2}{\mathrm{a}^2}\epsilon_1 + \frac{E_2}{E}\frac{a_2^2}{\mathrm{a}^2}\epsilon_2 + \frac{m}{2}\frac{a_1^5 - a_2^5}{\mathrm{a}^5}\right\}\left(\frac{1}{3} - \mu^2\right)$$

$$= \frac{E}{r} + \frac{E\mathrm{a}^2}{r^3}\left\{\epsilon - \frac{m}{2} - N\right\}\left(\frac{1}{3} - \mu^2\right)$$

$$\text{where } N = \frac{E_1}{E}\frac{a_1^2}{\mathrm{a}^2}\epsilon_1 - \frac{E_2}{E}\frac{a_2^2}{\mathrm{a}^2}\epsilon_2 - \frac{m}{2}\frac{a_1^5 - a_2^5}{\mathrm{a}^5},$$

and, as before, $3N$ is the whole decrease which would thus be

produced in gravity from the equator to the poles by this change of distribution of the mass. We shall calculate N for the second, third, and fourth shells.

$$N = \frac{1}{294}\left(\frac{0.6628}{1.09}\frac{16}{25} - \frac{0.3371}{1.16}\frac{9}{25}\right) - \frac{1}{578}\frac{1024-243}{3125},$$

$$\frac{1}{294}\left(\frac{0.3371}{1.16}\frac{9}{25} - \frac{0.1134}{1.20}\frac{4}{25}\right) - \frac{1}{578}\frac{243-32}{3125},$$

$$\frac{1}{294}\left(\frac{0.1134}{1.20}\frac{4}{25} - \frac{0.0153}{1.23}\frac{1}{25}\right) - \frac{1}{578}\frac{32-1}{3125},$$

$$= 0.0005354,\ 0.0001876,\ 0.0000322.$$

The ratios these, multiplied by 3, bear to the actual increase of gravity are

0.310, 0.167, 0.019.

And therefore the number of beats gained at Spitzbergen upon the pendulum at the equator would be 214 multiplied by these fractions, or 66, 23, and 4. The first and second of these might be detected, though not the third. This calculation shows that a comparatively small change in the form of the strata would have a very perceptible influence upon the pendulum. The effect is greater than is produced in the former case, by a mere change in densities without altering the form of the strata: and this shows the important effect which the bulging of the strata has on the pendulum. We might perhaps conceive the *external* surface of an irregular mass revolving round a fixed axis assuming, after an enormous period, a generally spheroidal form, because the perpetual weathering of the surface would set free parts of the solid materials, which with the fluids would arrange themselves according to fluid principles. But the *interior* parts could not thus arrange themselves, unless they had at one time been fluid or semi-fluid, so as to partake of that bulging form about the equator of each stratum which the motion of rotation tends to produce.

128. *Third re-arrangement.* The following is perhaps a still better hypothetical arrangement of the earth's mass with

a view to testing the effect of a departure from the fluid arrangement. Suppose that the earth is a solid mass which, as described above, has acquired its external spheroidal form by the action of time; and imagine its mass to be made up of a homogeneous spheroid of the earth's present form, but of the density only of the surface, with the remainder of the mass distributed anyhow in spherical shells around the centre.

The density of the surface is half the mean density of the earth; hence the mass of the homogeneous spheroid will be half the whole mass;

$$\therefore\ U=\frac{\frac{1}{2}E}{r}+\frac{\frac{1}{2}Ea^2}{r^3}\left(\epsilon-\frac{m}{2}\right)\left(\frac{1}{3}-\mu^2\right)+\frac{\frac{1}{2}E}{r}$$

$$=\frac{E}{r}+\frac{1}{2}\frac{Ea^2}{r^3}\left(\epsilon-\frac{m}{2}\right)\left(\frac{1}{3}-\mu^2\right),$$

and the consequent increase of gravity between the equator and the poles will

$$=\frac{3}{2}\left(\epsilon-\frac{1}{2}m\right),$$

or, when compared with the actual increase of gravity,

$$=\frac{3}{2}\left(\frac{1}{294}-\frac{1}{578}\right)\div 0{\cdot}0051828=0{\cdot}48.$$

This is nearly half the actual increase. Hence if this were the actual distribution, the gain of the pendulum over its rate at the equator would everywhere be only about half what experiment makes it to be.

Experiment shows no marked deviation from a regular increase, varying as the change in the square of the latitude, in the rates of the pendulum in passing from the equator towards the poles. Hence the excess of matter above the homogeneous spheroid cannot be distributed irregularly. We have supposed it to be distributed in spherical shells, and the change on the pendulum would be, as we have shown, very great, and would be very perceptible indeed. Any departure from the spherical form, not towards the oblate spheroids required by the fluid-theory, but in the opposite direction,

would produce a result still more discordant with experiment; whereas every approach in the distribution towards those spheroids will bring the calculation into nearer accordance with fact.

129. COR. We may find the effect of a large departure from regularity in the mass in the following manner. Suppose there is a preponderance of matter the effect of which may be represented by a spherical mass m, the distance of the centre of which from the centre is $c, = x \,.\, \mathrm{a}$ suppose. Then the difference of gravity in consequence of this at the two points nearest and furthest off from this mass

$$=\frac{m}{(\mathrm{a}-c)^2}-\frac{m}{(\mathrm{a}+c)^2}=\frac{4c\mathrm{a}m}{(\mathrm{a}^2-c^2)^2}=\frac{4xm}{\mathrm{a}^2(1-x^2)^2},$$

the ratio of this to gravity

$$=\frac{m}{E}\frac{4x}{(1-x^2)^2}.$$

If b be the number of beats lost or gained by a seconds' pendulum between the two places in 24 hours

$$\frac{2b}{24\times 60\times 60}=\frac{m}{E}\frac{4x}{(1-x^2)^2};$$

$$\therefore \frac{m}{E} \text{ or } \frac{e^3}{\mathrm{a}^3}=x\left(\frac{1}{x}-x\right)^2\times 0{\cdot}00000579b,$$

where e is the radius of a sphere of which the density is the mean density of the earth and the mass $= m$.

Ex. 1. Suppose $b=10$, $x=0.75$, then $e=96$ miles. That is, a mass of only radius 96 miles and only fourteen millionths and a half of the earth's mass, and as far down as 1000 miles from the surface, will have a perceptible influence upon the pendulum.

Ex. 2. If the depth below the surface be 500 miles the radius of the disturbing mass will be only 62 miles, and the mass three millionths and three-quarters of the earth's mass.

Ex. 3. If $x = 0.25$, $e = 236$ miles and m is 1-5000th part of the earth's mass, and 3000 miles below the surface; and yet in each of these cases the effect is the same as before.

Accurate pendulum experiments all over the world must bring to light such masses, if they exist. None have as yet been detected.

130. We have now completed the investigation of the fluid theory of the earth's figure. It teaches us (1) that gravity must everywhere be perpendicular to the surface (Art. 108); (2) that the exterior surface and the surfaces of the strata are all concentric spheroids, with the axis of each coincident with the axis of the earth (Art. 110).

These spheroids are of a definite form, depending upon the velocity of rotation and the law of density, as we have seen in Arts. 112, 119. These spheroids so determined are called "spheroids of equilibrium," because they are the forms which the mass assumes when in equilibrium if it be fluid throughout. This term is used whether the mass has subsequently become solid or not, and refers solely to the form, not the condition. And in general when we say a body has a "surface of equilibrium," we mean that the surface though solid would retain its form if it became fluid, all other things remaining the same.

131. We cannot close this investigation without calling attention to the remarkably near coincidence between the ellipticity of the surface we have derived from the fluid theory with that which geodetic measures produce. The fluid theory makes it $\frac{1}{293}$, see Art. 118; whereas, as will be seen in the Third Chapter, geodetic measurement makes it $\frac{1}{295}$.

§ 3. *The thickness of the Earth's Crust.*

132. Even if the earth ever has been a fluid mass, it is evidently not wholly fluid now; this the existence of continents and ocean-beds attests. It is a question how

far down this solidity prevails. It is supposed by some that the crust may be of comparatively small thickness, so thin as to be influenced in its position of elevation and depression by the fluid mass below, on which it is in fact imagined to float.

The late Mr W. Hopkins of Cambridge endeavoured to ascertain how far the interior of the Earth may at present be fluid, by calculating the value of the Precession of the Equinoxes upon the supposition of the mass being a spheroidal shell of heterogeneous matter, enclosing a heterogeneous fluid mass, consisting of strata increasing according to the law we have used. In three memoirs in the *Philosophical Transactions* of 1839, 1840, and 1842, he entered upon a complete investigation of this subject. We will give the evidence upon which his conclusion rests that the crust is very thick.

PROP. *To trace the argument drawn from Precession to show that the crust is of considerable thickness.*

133. Mr Hopkins deduced the following formula (in which we have changed the notation to suit the present treatise,)

$$\frac{P-P'}{P}=1-\frac{\epsilon'}{\epsilon}-\frac{\displaystyle\int_a^{\mathrm{a}}\rho'\,\frac{d\,.\,a'^5(\epsilon''-\epsilon')}{da'}\,da'}{\displaystyle 2\epsilon a^3\int_0^a\rho' a'da'+\epsilon\int_a^{\mathrm{a}}\rho'\,\frac{d\,.\,a'^5}{da'}\,da'},$$

where P is the precession of the equinoxes of a homogeneous spheroid of ellipticity ϵ, which by calculation $=57''$ nearly if $\epsilon=\frac{1}{300}$; P' is the precession of the heterogeneous shell, the outer and inner ellipticities being ϵ and ϵ', which is taken to represent the earth's crust; this $=50''.1$ by observation.

The success of the calculation depends upon a remarkable result at which he arrived, that the precession caused by the disturbing forces in a homogeneous shell filled with homogeneous fluid, in which the ellipticities of the inner and outer surfaces are the same, is the same whatever the thickness of the shell. It is therefore the same for a spheroid solid to the

centre. The formula above given is the relation of the amounts of precession in two shells, one heterogeneous and the other homogeneous; and, as the thickness is the quantity sought, neither of these amounts could be calculated, and therefore the relation expressed in the above formula would be of no avail. But in consequence of the property that the precession of the shell, when it and the fluid are homogeneous, is the same as that of the solid spheroid, this difficulty is overcome; and P can be calculated without knowing the thickness, and therefore P' will be known. In fact, from the above data we have

$$\frac{P'}{P} = \frac{50}{57} = \frac{7}{8}.$$

We have shown (Art. 114) that the strata decrease in ellipticity in passing downwards: hence $\epsilon'' - \epsilon'$ is never negative, and the fraction on the right hand in the above formula is never negative, and is never so large as unity: let it $= \beta$. Hence

$$\frac{\epsilon'}{\epsilon} = \frac{7}{8} - \beta, \text{ or } \epsilon' \text{ is less than } \frac{7}{8}\epsilon;$$

and therefore, because the ellipticity decreases in descending, the thickness must be greater than would correspond with an ellipticity of the inner surface of the shell equal to 7-8ths of that of the outer surface.

If solidification took place solely from pressure, the surfaces of equal density would be surfaces of equal degrees of solidity. If we use the formula for finding ϵ in Art. 117, and make $qa = 150°$, and the mean density $= 2.4225$ times the superficial density (from Mr Airy's Harton calculation), then if $\epsilon' = \frac{7}{8}\epsilon$ in the formula of Art. 117, we have, after reduction, $a = \frac{3}{4}$a, or the thickness equal to one-fourth of the radius, or 1000 miles. If a smaller ratio of densities is used than 2.4225, the thickness is greater. Mr Hopkins showed also that a ratio a little larger than 3 makes the thickness 1-5th of the radius: but this ratio is too large. The ratio generally used is about 2.2 or even 2.

But solidification depends upon temperature, as well as upon pressure. In his third memoir (*Phil. Trans.* 1842), Mr Hopkins showed that the isothermal surfaces increase in ellipticity in passing downwards. If temperature alone regulated the solidification, these surfaces would be the surfaces of equal solidity. But since both pressure and temperature have their effects, the ellipticities of the surfaces of equal solidity must lie between those of the isothermal and the equi-dense surfaces. Hence the surface of equal solidity at any depth will be more elliptic than the surface of equal density at that depth: and therefore the inner surface of the solid shell, of which the ellipticity is $\frac{7}{8}\epsilon$, must be at a depth corresponding to a stratum of equal density of smaller ellipticity than $\frac{7}{8}\epsilon$, that is, at a greater depth than 1000 miles.

In the above reasoning β has been neglected. If its value be used, it strengthens the argument for a greater thickness than 1000 miles.

We may, therefore, safely conclude that 1000 miles is the least thickness of the solid crust. In the calculation it has been assumed that the transition from the solid shell to the fluid nucleus is abrupt. This will hardly be the case. The above result will therefore apply to the *effective* surface, lying near the really solid shell. But in consequence of the tendency, as shown above, of every cause being to prove that the crust is really thicker than 1000 miles, we may safely take this to be its least limit*.

134. In 1868 M. Delaunay of Paris read a paper before the French Academy of Sciences calling in question the soundness of Mr Hopkins' reasoning, on the ground that the motion of precession and nutation is so slow, that the fluid, owing to friction and viscidity, would partake of the same motion

* It will be observed that the ellipticity of the surface of the earth is in these pages always denoted by the symbol ϵ, the mean radius of the surface being a. Owing, however, to the want of variety of Greek type this same symbol has occasionally been used to denote the ellipticity of an internal stratum of mean radius a. No confusion will arise from this, as the meaning of the symbol in each case is sufficiently obvious.

exactly as the crust*. But it is not merely the resultant motion which we have to consider. The particles of the earth's mass have more to do than to move slowly in producing precession: they have to bear a *strain*, which no fluid, though viscous, could sustain and transmit. As the question is important we will enter upon it more in detail; omitting, however, calculations which are irrelevant to this treatise, and merely pointing out how they should be made, leaving it to the student to fill them in; which he will find no real difficulty in doing, if he is well acquainted with the Integral Calculus and the Higher Mechanics.

PROP. *To show that the solid crust cannot be thin.*

135. Suppose the earth to have a solid crust, to some extent elastic, and filled with heterogenous fluid. We shall suppose that the disturbing forces draw out the crust on opposite sides, causing an elevation superimposed on the mean oblate spheroid; and that the elevation of the inner surface is less in the ratio of the mean radii. There are three causes which would tend to disturb the axis of the crust: (1) the disturbing attraction of the sun and moon on the crust itself; (2) the same on the fluid, producing a pressure against the crust; (3) the additional centrifugal force produced by the solid crust being drawn out into a new shape. It can be shown by Mechanics that the sun, from each of these causes, would produce a moment round that equatorial diameter of the earth, which is at right angles to the line joining the earth and sun: and these, added together, produce the solar precession and nutation. So for the moon; the axis of moments being in that case at right angles to the line joining the earth and moon. In the calculation the fluid, at the moment under consideration, is supposed to be arranged as to density according to M. Delaunay's idea; viz. exactly as if it and the crust had been hitherto one solid mass. The result shows that this state cannot possibly continue. The final formula is, the Precession or P

* The paper is translated in the *Geological Magazine*, November, 1868, p. 507.

$$= \frac{16225''6 \left\{ \int_a^{\text{a}} \rho \frac{d.a^5\epsilon}{da} da + \frac{25}{64} a^3 \epsilon \int_0^a \rho a da - 13377 \left(D\text{a}^4 h - D' a^{4\prime} h \right) \right\}}{\int_a^{\text{a}} \rho \frac{d.a^5}{da} da},$$

a, a are the mean radii; D, D' the densities; h, h' the elevations of the outer and inner surfaces owing to elasticity. The fluidity is here supposed perfect: but an allowance will be made in the application of the formula for viscosity. The three terms in P arise severally from the three causes above enumerated. There would be another term, similar to the second, for the altered shape of the fluid mass arising from the supposed flexibility of the crust. But it would be quite evanescent. We will examine what this formula teaches us.

I. If the earth be solid to the centre $a = 0$, $h' = 0$, and the second term (arising from fluid pressure) disappears. If also the earth is rigid $h = 0$, and (see Art. 146),

$$P = 16225''.6 \left(\int_0^{\text{a}} \rho \frac{d.a^5 e}{da} da \div \int_0^{\text{a}} \rho \frac{d.a^5}{da} da = \frac{C - A}{C} = 0.003312 \right).$$

This equals $53''.74$; $\epsilon = 1 \div 294$, $m = 1 \div 289$. Observation makes the precession $50''.1$. Hence h must have a minute value, and the mass must be slightly elastic. The value of h which will make the calculated and observed results agree is

$$h = \frac{1}{13377 D\text{a}^4} \left(\int_0^{\text{a}} \rho \frac{d.a^5 \epsilon}{da} da - \frac{50.1}{16225.6} \int_0^{\text{a}} \rho \frac{d.a^5}{da} da \right),$$

by Art. 146,

$$= \frac{0.000224}{13377 D\text{a}^4} \int_0^{\text{a}} \rho \frac{d.a^5}{da} da = \left(\frac{2}{z} + 1 - \frac{6}{q^2 \text{a}^2} \right) \frac{0.00112 \text{a} z}{13377 q^2 \text{a}^2}.$$

This equals 0.6 foot; by Art. 119. This is a small quantity; it must produce some effect on the tides, though not perhaps so large as to be observable. But more of this presently.

II. Suppose there is a solid crust with fluid within. Now $\int_a^{\text{a}} \rho \frac{d.a^5 \epsilon}{da} da \div \int_a^{\text{a}} \rho \frac{d.a^5}{da} da$ is larger the larger the limiting

values of ϵ. Hence the first term in P is larger for a crust than for a solid earth; and is more so the thinner the crust is. In fact, we have shown above, that for a solid earth it is 0.003312; for a crust of infinitesimal thickness it is $\epsilon = 1 \div 294 = 0.003400$. Next, the action of the fluid pressure (shown in the second term) is still further to increase the Precession. Lastly, the effect of non-rigidity (shown in the third term) is to diminish the Precession. We will see whether, under any condition, this can be sufficient altogether to counteract the effect of the fluid pressure. Put the second and third terms together equal to zero, making $h' \div h = a \div \mathrm{a}$, and using the ordinary law of density. Then

$$h = \frac{\mathrm{a}^2 a^3 (1 - \cos qa)\,\epsilon}{34245 q\mathrm{a}\,(\mathrm{a}^4 \sin q\mathrm{a} - a^4 \sin qa)}.$$

This is larger the larger a is. By help of Art. 119 we can apply this formula to various cases.

Suppose the thickness is 800 miles, or 1-5th of the radius. Then $qa = 112^\circ\, 47'$, $\cos qa = -0.3781101$, $\sin qa = 0.9257606$; also $q\mathrm{a} = 140^\circ\, 59' = 2.4605$, $\sin q\mathrm{a} = 0.6295464$. Then $h =$ 19.85 feet; that is, it would require an elevation of as much as 20 feet in the outer surface of the crust, arising from elasticity and flexibility, altogether to counteract the effect of fluid-pressure on the crust in moving the axis. The effect of this would be seen in the tides, as pointed out by Sir Wm. Thomson, see Art. 138. The sea rises only 2.8 feet in the open ocean at spring tides, that is, only about 1-7th part of this elevation. Hence, even if we allow that the fluid mass, owing to viscosity, produces pressure against the crust, under the action of the sun and moon, equal to only 1-7th part of its pressure when free from viscosity, yet would it require such a degree of elasticity and flexibility in the crust to counteract that effect as would entirely destroy the tides.

The case is, of course, still stronger if we take the crust only 100 miles thick. Then $a \div \mathrm{a} = 39 \div 40 = 0.975$, $qa = 137^\circ\, 28'$, $\cos qa = -0.7368842$, $\sin qa = 0.6760190$; and h is about 74 feet; that is, about 26 times the tide at springs. Hence, if we suppose, the crust being thus thin, that owing to viscosity, the fluid has only 1-26th part of its effect in producing pressure against the crust under the action of the sun and

moon, yet to counteract its effect the crust must have such a degree of pliability as would entirely destroy the tides.

In both these cases, moreover, the Precession of the solid crust, even after elasticity is allowed for to such an extent as would destroy the tides, would be greater even than $53''.74$ (the value of the first term of P for a rigid earth), whereas the observed value is $50''.1$. If therefore at any moment the crust and the fluid are moving alike, as one solid, this cannot continue, the crust would begin to move quicker.

It must be said, then, that these results clearly show, that the crust must be much thicker than 800 miles; as otherwise, in spite of an extravagant allowance for the effect of viscosity in counteracting pressure, the tides would not exist, a misfortune from which the world is happily not suffering.

136. Professors Hennessy, Haughton and Sir W. Thomson have written upon this subject: see *Phil. Trans.* 1851, *Transactions of the Royal Irish Academy*, 1852, and *Phil. Trans.* 1862. The first makes the thickness lie between 18 and 600 miles. But in his calculation he assumes that the shell is so rigid as to resist, without change of form, the internal pressure which arises from the inner surface ceasing to be one of fluid equilibrium: an assumption which cannot be considered admissible. Moreover he supposes that in cooling the outer shell will contract less than the fluid nucleus; which can hardly be true.

Mr Haughton's investigation is simply a problem of densities, and determines nothing whatever regarding the ratio of the solid to the fluid parts of the Earth. (See *Philosophical Magazine*, Sept. 1860.)

Sir W. Thomson, in his paper on the "Rigidity of the Earth," confirms Mr Hopkins' result, as will appear from the investigation with which this Chapter closes.

In a previous edition we gave an argument in favour of a great thickness, arising from the tendency of the weight of the enormous mass of the Himalaya mountains to break down the crust if it is thin; and of the fluid mass below extensive ocean-beds to burst them up under the same circumstances. Subsequent calculations, however, on another subject have shown that the crust below the mountains must be of less

density, and that below ocean-beds of greater density than the average, so as to produce a very considerable compensation. If this is the case the downward pressure in the first instance and the upward pressure in the second might be too feeble to produce fracture even if the crust be thin. (See Arts. 192—196.)

We will conclude this Chapter by giving some account of Professor W. Thomson's remarkable investigation regarding the "Rigidity of the Earth," in the first place making a calculation which will be of use. He accepts the result of Mr Hopkins' calculation, and indeed thinks that it might have been pressed further. His own investigations, an abstract of which we now give, point out that the earth's mass must possess such a degree of rigidity as to be altogether inconsistent with a crust of moderate thickness.

PROP. *A mass in the form of a prolate spheroid of small ellipticity consists partly of a nucleus of spherical shells concentric with the spheroid, the remaining portion of the spheroid being homogeneous: to find the form of the level surface, or surface at every point of which the resultant attraction of the whole mass acts in the normal, passing through the equator of the spheroid.*

137. Let a and $c = a(1+\epsilon)$ be the semi-axes, suppose a sphere of radius a to be inscribed touching the spheroid in its equator: ρ the density of the outer part of the spheroid, $N.\rho$ the mean density of the mass within the inscribed sphere. We may conceive the mass made up of a homogeneous prolate spheroid of density ρ, and a concentric sphere of radius a and mean density $(N-1)\rho$. Then by Art. 14, the attractions of the homogeneous spheroid parallel to the axis on a point xyz are

$$\frac{4}{3}\pi\rho\left(1+\frac{2}{5}\epsilon\right)x,\quad \frac{4}{3}\pi\rho\left(1+\frac{2}{5}\epsilon\right)y,\quad \frac{4}{3}\pi\rho\left(1-\frac{4}{5}\epsilon\right)z,$$

and the attractions of the sphere

$$\frac{4}{3}\pi\rho\,(N-1)\,\frac{a^3}{r^3}\,x,\quad \frac{4}{3}\pi\rho\,(N-1)\,\frac{a^3}{r^3}\,y,\quad \frac{4}{3}\pi\rho\,(N-1)\,\frac{a^3}{r^3}\,z.$$

If X, Y, Z are the total attractions parallel to the axes at

the point xyz, then, that their resultant at that point may act in the normal, we must have

$$Xdx + Ydy + Zdz = 0,$$

which, by substitution, integration and division, gives

$$\left(1 + \frac{2}{5}\epsilon\right)(x^2 + y^2) + \left(1 - \frac{4}{5}\epsilon\right)z^2 - 2(N-1)\frac{a^3}{r} = \text{const.},$$

$$\text{or } \left(1 + \frac{2}{5}\epsilon - \frac{6}{5}\epsilon \sin^2 l\right) r^2 - 2(N-1)\frac{a^3}{r} = \text{const. } (l = \text{latitude}).$$

Suppose $r = a(1 + \epsilon' \sin^2 l)$, which can evidently be made to satisfy this equation, is the equation to the level surface through the equator. By substituting for r and equating the constant parts and also the parts depending on $\sin^2 l$, the latter leads to

$$2\epsilon' - \frac{6}{5}\epsilon + 2(N-1)\epsilon' = 0,$$

$$\therefore \epsilon' = \frac{3}{5N}\epsilon,$$

which gives the form of the level surface.

PROP. *To explain how the Tides can be used as a test of the degree of Rigidity of the Earth's mass.*

138. It is necessary to premise, that, in an elaborate and difficult investigation on the deformation of a spheroidal elastic mass acted upon by external forces, Professor Thomson has deduced the following formula (*Phil. Trans.* 1863, p. 574),

$$h = \frac{h'}{1 + \dfrac{19}{2}\dfrac{n}{gwr}},$$

where h' and h are the polar elevations of a prolate spheroid drawn out from a spherical figure by an external force (resembling the combined action of the sun and moon in raising the tidal wave), the spheroid being considered a homogeneous mass of incompressible *fluid* in the first instance, and of incompressible *solid* matter in the second: in both cases the total mass is equal to the mass of the earth; w denotes

the mass of a unit of volume, that is, the density; r the radius of the globe, and n the rigidity of the solid substance, that is, $n.k$ is the force necessary to draw a unit of section of the solid through an extremely small space k.

Now suppose the earth an absolutely rigid sphere, and H the polar elevation of the prolate spheroid, or level surface, which is everywhere perpendicular to the resultant action of the sun and moon, which in the actual ocean produces the tidal wave. Then H is the height of high above low water, in the equilibrium theory of the tides, of an ocean of infinitely small density covering a rigid earth.

Suppose instead of the above that the earth is covered with an ocean, the earth still being a perfectly rigid sphere, and its mean density N times that of the ocean. Let H' be the polar elevation of the prolate spheroid into which the sun and moon draw the ocean. Then, by Art. 137, the terrestrial gravitation level would be disturbed by this cause, from the spherical surface to a spheroidal surface of which the polar elevation is $(3 \div 5N)\,H'$, by the attraction of the ocean in its altered form. The polar elevation in the level surface, before noticed as produced by the direct action of the sun and moon, must be added to this to give the polar elevation of the actual equilibrium level of the free surface of the ocean. Hence

$$H' = H + \frac{3}{5N}H', \quad \text{or } H' = \frac{H}{1 - \frac{3}{5N}}.$$

It will be observed that h' and H' are similar quantities, but for oceans of different density: when $N = 1$, $h' = H'$, and this $= \frac{5}{2}H$, and

$$\therefore\ h = \frac{5}{2}\frac{H}{1 + \frac{19}{2}\frac{n}{wgr}}.$$

We have thus far been considering the earth to be absolutely rigid. If its solid mass is drawn up from a spherical form to a polar elevation h by the sun and moon, the attraction of the protuberant mass will change the gravitation level from a sphere to a prolate spheroid of polar elevation

$(3 \div 5)\,h$: and this as before should be added to H to find the whole effect of the sun and moon in changing the gravitation level. The sum will be the absolute tidal elevation above, the sphere, of an ocean of infinitely small density covering the elastic globe. By subtracting h, the tidal elevation of the solid itself, we have the difference between high and low water

$$= H - \frac{2}{5}h, \quad = \frac{\frac{19}{2}\frac{n}{gwr}H}{1 + \frac{19}{2}\frac{n}{gwr}}.$$

If the earth be perfectly rigid, n is infinite, and this expression becomes H, as it ought to do. For iron or steel $n = 501 \times 10^6$, the unit of mass being 1 lb., the unit of space 1 foot. This makes the above expression for the height of the tide equal to $0.59H$ or about three-fifths of H. For glass $n = 2160000 \times 144 \times 32.2$, which makes the tide two-ninths of H.

"Imperfect," Professor Thomson remarks, "as the comparison between theory and observation as to the absolute height of the tides has been hitherto, it is scarcely possible to believe that the height is in reality only two-ninths of what it would be if, as has hitherto been universally assumed in tidal investigations, the earth were perfectly rigid. It seems therefore nearly certain, with no other evidence than is afforded by the tides, that the tidal effective rigidity of the earth must be greater than that of glass.

PROP. *To explain how Precession can be used as a test of the Earth's Rigidity.*

139. Conceive the earth to be a fluid mass revolving in a day about its axis, and drawn by centrifugal force into a spheroidal figure having the same amount of protuberant matter as the earth actually has. Suppose also that the combined action of the sun and moon produces a tidal wave on each side of the earth which is superimposed upon the spheroid of revolution. As the earth in this case would have no rigidity whatever, it would therefore have no precession. Conceive things to remain the same, except that the fluid

becomes an elastic solid, yielding as the fluid to the varying influence of the sun and moon, so as to produce a tidal wave superinduced as before on the spheroid of revolution. This mass would still have no precession. As the mass is now solid, though elastic, the sun and moon, by attracting the nearer equatorial parts more, and the further equatorial parts less, than the centre, would have a tendency to cause it to rotate round an axis in the equator and produce precession. But as, in the case supposed, no precession takes place, this tendency of the sun and moon to produce precession must be exactly counterbalanced by some opposite tendency: that tendency is the effect of the centrifugal force on the protuberant parts of the tidal wave drawn up on the solid by the sun and moon. This effect, therefore, of the solid tidal wave would in amount exactly equal the actual precessional motion of the earth, on the hypothesis of the earth's mass being perfectly rigid, though it would act in the opposite direction.

Now by the last Article it appears, that if the earth have the rigidity of steel or glass the tides would be reduced to three-fifths or two-ninths of their amount on the supposition that the earth's rigidity is perfect; that is, the deformation of the solid parts beneath the ocean would be two-fifths or seven-ninths of that amount. The result would be that two-fifths or seven-ninths of the precession caused in a rigid earth would be balanced if the mass have only the rigidity of steel or glass. As no such effect is detected by observation it must be presumed that the rigidity of the earth is decidedly very great.

"The close agreement," Professor Thomson remarks, "between the observed amounts of precession and nutation and the results of theory on the hypothesis of perfect rigidity, renders it impossible to believe that there is enough of elastic yielding to influence the phenomena to any considerable extent. It is worthy of remark, however, that in general the theoretical estimates of the amount of precession have been somewhat above the true amount demonstrated by observation. It seems not altogether improbable that this discrepancy is genuine, and is to be explained by some small amount of deformation experienced by the solid parts of the earth, under lunar and solar influence."

CHAPTER II.

THE FIGURE OF THE EARTH DETERMINED FROM PENDULUM EXPERIMENTS, THE MOON'S MOTION, AND THE PRECESSION OF THE EQUINOXES.

140. Upon the hypothesis of the earth being a fluid mass it was shown by Clairaut, in his celebrated work, *Figure de la Terre*, published in 1743, that the increase of gravity in passing from the equator to the poles varies as the square of the sine of the latitude, and that a certain relation must necessarily subsist between the ellipticity and the amount of gravity, a relation which has been ever since known as *Clairaut's Theorem*. Laplace demonstrated the same, on the simpler hypothesis of the surface only being a surface of equilibrium, and the interior being solid or fluid, but consisting of strata nearly spherical. Professor Stokes has done the same without making any assumption whatever regarding the constitution of the interior of the mass, but assuming only that the surface is a spheroid of equilibrium of small ellipticity.

Clairaut's Theorem is valuable as it enables us to determine the ellipticity by means of pendulum oscillations, the times of which measure the force of gravity at the several stations where experiments are made. The following is taken from Professor Stokes' demonstration in the *Cambridge Philosophical Transactions* for 1849.

PROP. *To obtain the potential of the earth's mass for an external point, assuming only that the surface is a spheroid of equilibrium of small ellipticity.*

141. Let V be the potential of the mass. Then because the surface is a surface of equilibrium (see Art. 108),

$$\text{const.} = V + \tfrac{1}{2}\omega^2(1-\mu^2)\,r^2.$$

By Art. 25 we have, for an external point,

$$r\frac{d^2 . rV}{dr^2} + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2V}{d\omega^2} = 0.$$

Let V be expanded in a series of Laplace's Functions,

$$V_0 + V_1 + \ldots + V_i + \ldots$$

Then since the above equation is linear with respect to V, and a series of Laplace's Functions cannot equal zero unless the Functions are separately zero (see Art. 38), we have, by substituting the above series for V and remembering the condition given by Laplace's Equation,

$$r\frac{d^2 . rV_i}{dr^2} - i(i+1)\,V_i = 0.$$

Multiply by r^{-i+1} and integrate;

$$\therefore\; r^{-i}\frac{d(rV_i)}{dr} + ir^{-i-1}(rV_i) = \text{const.} = (2i+1)\,Z_i \text{ suppose.}$$

Multiply by r^{2i} and integrate;

$$\therefore\; r^i(rV_i) = Z_i r^{2i+1} + W_i, \text{ or } V_i = \frac{W_i}{r^{i+1}} + Z_i r^i;$$

where W_i and Z_i are independent of r;

$$\therefore\; V = \frac{W_0}{r} + \frac{W_1}{r^2} + \frac{W_2}{r^3} + \ldots + Z_0 + rZ_1 + r^2 Z_2 + \ldots$$

Now V evidently vanishes, from its very definition, when r is infinite. Hence $Z_0 = 0$, $Z_1 = 0$, $Z_2 = 0 \ldots$

$$\therefore\; V = \frac{W_0}{r} + \frac{W_1}{r^2} + \frac{W_2}{r^3} + \ldots$$

If there were no centrifugal force and the surface of equilibrium were exactly spherical instead of being nearly so, the force at the surface, and therefore V, would be the same in all directions from the centre, and W_1, $W_2 \ldots$ would be zero. Hence in our case W_1, $W_2 \ldots$ must be all small quantities of the first and higher orders.

Substitute for V in the equation of equilibrium and put

$$r = \mathrm{a}\left\{1 + \epsilon\left(\frac{1}{3} - \mu^2\right)\right\}, \text{ (Art. 110)};$$

$$\therefore \text{const.} = \frac{W_0}{\mathrm{a}}\left\{1 - \epsilon\left(\frac{1}{3} - \mu^2\right)\right\} + \frac{W_1}{\mathrm{a}^2} + \frac{W_2}{\mathrm{a}^3} + \ldots + \frac{1}{2}w^2\mathrm{a}^2(1 - \mu^2).$$

Equate the sums of Laplace's Functions of the same order to zero;

$$\therefore\ W_0 = \text{const.} - \frac{1}{3}w^2\mathrm{a}^3, \quad W_1 = 0,$$

$$W_2 = \left(W_0\epsilon - \frac{1}{2}w^2\mathrm{a}^3\right)\mathrm{a}^2\left(\frac{1}{3} - \mu^2\right), \quad W_3 = 0, \quad W_4 = 0 \ldots$$

$$\therefore\ V = \frac{W_0}{r} + \left(W_0\epsilon - \frac{1}{2}w^2\mathrm{a}^3\right)\frac{\mathrm{a}^2}{r^3}\left(\frac{1}{3} - \mu^2\right).$$

W_0 is evidently equal to the mass, because as r becomes infinitely great the second term vanishes with reference to the first, and we know that in that case the value of the potential must be the mass divided by the distance. Let $W_0 = E$. Also put $m = w^2 . \mathrm{a}^3 \div E$, as in Art. 108;

$$\therefore\ V = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right)\frac{E\mathrm{a}^2}{r^3}\left(\frac{1}{3} - \mu^2\right),$$

the formula required. It is precisely the same as that obtained in Art. 121 on the fluid theory.

We may obtain from this formula, as in Art. 122, the following expression for gravity along the earth's surface,

$$g = G\left\{1 + \left(\frac{5}{2}m - \epsilon\right)\sin^2 l\right),$$

and at once deduce Clairaut's theorem from it as before; the proof of which will therefore be independent of any theory regarding the arrangement of the mass.

We shall in this Chapter make three uses of this formula in order to obtain three independent measures of the earth's ellipticity, by applying it to the results of Pendulum experi-

ments, of the Moon's motion in latitude, and of the Precession of the Equinoxes[*].

142. When this formula is applied to the several stations where pendulum experiments have been made, discrepancies are brought to light as will be seen by referring to the Table in Art. 123, which evidently arise, not from any mistake in the theory, but from the irregularities of the surface of the earth. Mr Airy, in his article on the Figure of the Earth, written in 1830 (*Encyc. Met.*), discussed all the data which had been then obtained in various places, and came to the following results: (1) that gravity appears to be greater on islands than on continents; (2) that gravity is greater in high north latitudes, less in middle latitudes than the formula gives it, but is pretty nearly the same about the equator; (3) that gravity does not appear to vary with the longitude alone, nor the hemisphere.

Professor Stokes, in his paper in the *Cambridge Philosophical Transactions* for 1849 already referred to, has fully discussed the various causes of disturbance, and has satisfactorily explained the anomalies (1) and (2) deduced by Mr Airy, twenty years earlier, from the experiments. He has also shown, that when allowance is made for the irregularities of the earth's surface, the ellipticity comes out somewhat smaller than it otherwise would.

The chief sources of error are the elevation of the station above the sea-level, and the excess or defect of matter in table-lands or the sea.

The value of gravity obtained by pendulum experiments must be reduced to the standard of the sea-level, and corrected for that level in the way explained in Art. 64. But the sea-level, owing to local attraction, evidently rises higher in continents in passing from the sea. Hence gravity obtained from continental experiments will be too small, because it is corrected for a surface too distant from the centre of the earth. This explains why gravity appears to be less on continents

* In the *Cambridge and Dublin Mathematical Journal*, Nos. xx. xxi. May and November, 1849, Professor Stokes has demonstrated the same without any use of Laplace's Functions. But the demonstration is much longer than that given above.

than on islands. The same explanation meets the second anomaly pointed out by Mr Airy. In the middle latitudes the places where experiments were made are all continental. If this is corrected for, no doubt the deduced ellipticity will come out somewhat smaller, and therefore gravity in high latitudes, as deduced from the formula, no longer be in excess.

Mr Stokes remarks, that if the 49 stations where pendulum experiments were made are divided into two groups, an equatorial group containing the stations lying between latitudes 35° N. and 35° S., and a polar group containing the rest, it will be found that most if not all of the oceanic stations are contained in the former group, while the stations belonging to the latter are of a more continental character. Hence the observations will make gravity appear too great about the equator and too small about the poles, that is, they will on the whole make gravity vary too little from the equator to the poles; and since the variation depends on $\frac{5}{2}m-\epsilon$, the observations will be best satisfied by a value of ϵ which is too great. This is, in fact, precisely the result of the discussion; the value of ϵ which Mr Airy has obtained from pendulum experiments (0.003535) being, as stated, greater than that which he finds from the discussion of geodetic measures (0.003352).

In a former part of this treatise we have deduced formulæ for calculating the effect on gravity of the superficial irregularities of the earth when they are known. We are as yet unable to say what the exact amount of correction in consequence of these variations of the surface should be; we must therefore find the ellipticity without allowing for their influence.

PROP. *To find the ellipticity by Clairaut's Theorem and Pendulum experiments.*

143. By Clairaut's theorem

$$\frac{\text{increase of gravity from poles to equator}}{\text{gravity at equator}} + \text{ellipticity}$$

$$= \frac{5}{2} \times \text{ratio of centrifugal force to gravity, both at equator.}$$

But by Art. 123 the first ratio $=0.0051828$, and the second ratio $=0.0034602$ by Art. 100;

$$\therefore\ \epsilon = 0.0086505 - 0.0051828 = 0.0034677 = \frac{1}{288}.$$

The investigations of Professor Stokes referred to above show that this should be a little smaller.

144. There are two phenomena in Physical Astronomy which depend upon the ellipticity of the earth's figure and which Laplace used as tests of the measure obtained by other means. These are an inequality in the Moon's motion in latitude and the Precession of the Equinoxes.

PROP. *To find the effect of the ellipticity of the Earth's figure upon the Moon's motion in latitude.*

145. By the Planetary Theory (see Pratt's *Mechanical Philosophy,* second edition, p. 329; or Cheyne's *Planetary Theory,* p. 35),

$$\frac{di}{dt} = \frac{na}{(E+M)\,i}\frac{dR}{d\Omega}, \quad \frac{d\Omega}{dt} = -\frac{na}{(E+M)\,i}\frac{dR}{di},$$

where n is the mean motion of the moon about the earth, a the mean distance, E and M the masses, i the inclination of the moon's orbit to the ecliptic (the square of which is neglected), Ω the longitude of its node, R the disturbing function such that its differential coefficient with respect to any line drawn from the moon is the disturbing force acting on the moon in that direction, reckoned positive if acting on the side of the origin of the co-ordinates. If V be the potential of the earth with reference to the moon condensed into its centre, then $MV \div E$ will be the potential of the moon with reference to the earth; and in calculating the motion of the moon about the earth, we must imagine the earth reduced to rest by the moon's attraction being applied in an opposite direction to both the earth and moon. Hence the disturbing function R, which refers to the difference of attraction of the earth's mass as condensed into its centre and as arranged according to the fluid-law,

$$= \frac{E+M}{E}V - \frac{E+M}{r} = \left(\epsilon - \frac{m}{2}\right)\frac{(E+M)\,a^2}{r^3}\left(\frac{1}{3} - \mu^2\right),$$

r being the distance of the moon from the earth's centre. Let λ and θ be the latitude and longitude of the moon, ϵ' the epoch, ϖ the longitude of the perigee, I the obliquity of the ecliptic. Then, as λ and i are both small,

$$\tan\lambda = \tan i \sin(nt + \epsilon' - \Omega), \text{ or } \lambda = i\sin(nt + \epsilon' - \Omega).$$

$$\text{Also } \theta = nt + \epsilon' + 2e\sin(nt + \epsilon' - \varpi);$$

$$\therefore\ \mu = \cos\text{ (moon's north polar distance)}$$
$$= \sin I\cos\lambda\sin\theta + \cos I\sin\lambda$$
$$= \sin I\sin(nt + \epsilon') + i\cos I\sin(nt + \epsilon' - \Omega)$$
$$- e\sin I\sin\varpi + e\sin I\sin(2nt + 2\epsilon' - \varpi).$$

Substituting this in R, and preserving only the terms which are periodical and also independent of $nt + \epsilon'$, since these last go through their changes so rapidly as to neutralize their effects very quickly, we have

$$R = \frac{(E+M)\,a^2}{2r^3}\left(\epsilon - \frac{m}{2}\right) i\sin 2I\cos\Omega = (E+M)\,A\,.\,i\cos\Omega\,;$$

$$\therefore\ \frac{di}{dt} = -naA\sin\Omega,\ \ \frac{d\Omega}{dt} = -\frac{na}{i}A\cos\Omega.$$

Since the node on the whole retrogrades pretty steadily, we may put $\Omega = -ht$ on the second side of these equations, h being the mean regression. Hence δ being the symbol of variation in i and Ω,

$$\delta i = -\frac{na}{h}A\cos\Omega,\ \ \delta\Omega = +\frac{na}{hi}A\sin\Omega\,;$$

$$\therefore\ \delta\lambda = \sin(nt + \epsilon' - \Omega)\,\delta i - i\cos(nt + \epsilon' - \Omega)\,\delta\Omega$$

$$= -\frac{na}{h}A\sin(nt + \epsilon')$$

$$= -\frac{na^2}{2ha^3}\left(\epsilon - \frac{m}{2}\right)\sin 2I\sin(nt + \epsilon'),$$

putting $r = a$. Burg makes this term

$$= -8''\sin(nt + \epsilon')$$

by observation. Also $h = 0.0040217n$. Hence after all reductions

$$\epsilon - \frac{m}{2} = 0.0015474, \text{ and } \frac{m}{2} = \frac{1}{578} = 0.0017301;$$

$$\therefore \epsilon = 0.0032775 = \frac{1}{305} \text{ nearly.}$$

This result differs but slightly from the measure obtained by geodesy; it is a little too small. But considering the minuteness of the quantity to be determined, the result is remarkable.

PROP. *To determine the ellipticity of the Earth's figure from the amount of Precession of the Equinoxes.*

146. The Annual Precession

$$= \frac{C-A}{C} \frac{3n'}{n} \cos I \left(1 + \frac{n''^2}{n'^2} \frac{1 - \frac{3}{2}\sin^2 i}{1+\nu}\right) 180^\circ,$$

I = obliquity of the ecliptic $= 23^\circ\, 28'\, 18''$, i = inclination of Moon's orbit to ecliptic $= 5^\circ\, 8'\, 50''$, n and n' are the mean motions of the Earth round its axis and round the Sun, and their ratio $= 365.26$, n'' the mean motion of the Moon round the Earth $= 27.32$ days, ν = ratio of masses of Earth and Moon $= 75$. (See Pratt's *Mechanical Philosophy*, Second Edition, Art. 470: also, changing the notation, *Airy's Tracts*, Fourth Edition, p. 213, Arts. 36, 38.) Substituting the above quantities,

$$\text{Annual Precession} = 16225''.6 \frac{C-A}{C},$$

where A and C are the principal moments of inertia of the mass, the latter about the axis of revolution. To find these let xyz be the co-ordinates to any element of the mass, $r\theta\omega$ be the polar co-ordinates to the same. Then the mass of this element $= -\rho r^2 d\mu d\omega dr$, $\mu = \cos\theta$. Also

$$\frac{y^2+z^2}{r^2} = 1-(1-\mu^2)\cos^2\omega = \frac{2}{3} + \left\{\frac{1}{3} - (1-\mu^2)\cos^2\omega\right\},$$

$$\frac{x^2+z^2}{r^2} = \frac{2}{3} + \left\{\frac{1}{3} - (1-\mu^2)\sin^2\omega\right\}, \quad \frac{x^2+y^2}{r^2} = \frac{2}{3} + \left(\frac{1}{3} - \mu^2\right).$$

The terms are here arranged as Laplace's Functions. (See Art. 42, Ex. 4.)

$$\therefore\ C - A = \int_{-1}^{1}\int_0^{2\pi}\int_0^r \rho\,\{(x^2+y^2)-(y^2+z^2)\}\,r^2 d\mu d\omega dr$$

$$= \int_{-1}^{1}\int_0^{2\pi}\int_0^r \rho r^4 \left[\left(\frac{1}{3}-\mu^2\right) - \left\{\frac{1}{3} - (1-\mu^2)\cos^2\omega\right\}\right] d\mu d\omega dr.$$

Now r = radius of any stratum $= a\left\{1+\epsilon\left(\frac{1}{3}-\mu^2\right)\right\}$ (Art. 110);

$$\therefore\ \int_0^r \rho r^4 dr = \frac{1}{5}\int_0^a \rho \frac{d\,.\,r^5}{da}\,da$$

$$= \frac{1}{5}\int_0^a \rho \frac{d}{da}\left[a^5\left\{1+5\epsilon\left(\frac{1}{3}-\mu^2\right)\right\}\right] da$$

$$= \int_0^a \rho \left\{a^4 + \frac{d\,.\,a^5\epsilon}{da}\left(\frac{1}{3}-\mu^2\right)\right\} da$$

$$= \sigma(\mathrm{a}) + \psi(\mathrm{a})\left(\frac{1}{3}-\mu^2\right) \text{ suppose;}$$

$$\therefore\ C - A = \psi(\mathrm{a})\int_{-1}^{1}\int_0^{2\pi}\left(\frac{1}{3}-\mu^2\right)\{-\mu^2 + (1-\mu^2)\cos^2\omega\}\,d\mu d\omega,$$

$$= \pi\psi(\mathrm{a})\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)(1-3\mu^2)\,d\mu\,(\text{Art. } 29) = \frac{8\pi}{15}\psi(\mathrm{a}).$$

Also $C = \frac{8\pi}{3}\sigma(\mathrm{a})$, neglecting the small term $\psi(\mathrm{a})$.

Now $\quad \psi(\mathrm{a}) = \int_0^a \rho \frac{d\,.\,a^5\epsilon}{da}\,da = \frac{5}{3}\mathrm{a}^2\phi(\mathrm{a})\left(\epsilon - \frac{m}{2}\right)$

$$= \frac{5Q\mathrm{a}^2}{q^2}\sin q\mathrm{a}\left(\epsilon - \frac{m}{2}\right)z, \text{ by Arts. 113, 117.}$$

And putting $\rho = \frac{Q}{a} \sin qa$, and integrating by parts,

$$\sigma(\mathrm{a}) = \int_0^{\mathrm{a}} \rho a^4 da = Q \int_0^{\mathrm{a}} a^3 \sin qa da$$

$$= Q\left(-\frac{\mathrm{a}^3}{q} \cos q\mathrm{a} + \frac{3\mathrm{a}^2}{q^2} \sin q\mathrm{a} + \frac{6\mathrm{a}}{q^3} \cos q\mathrm{a} - \frac{6}{q^4} \sin q\mathrm{a}\right)$$

$$= \frac{Q\mathrm{a}^2}{q^2} \sin q\mathrm{a} \left\{3 - \frac{6}{q^2\mathrm{a}^2} - \left(1 - \frac{6}{q^2\mathrm{a}^2}\right) \frac{q\mathrm{a}}{\tan q\mathrm{a}}\right\}.$$

Hence substituting z, as before,

$$\frac{C-A}{C} = \frac{z}{2 + \left(1 - \frac{6}{q^2\mathrm{a}^2}\right) z} \left(\epsilon - \frac{m}{2}\right).$$

Substituting for $q\mathrm{a}$ and z, the values in Art. 118,

$$\frac{C-A}{C} = 1.98177 \left(\epsilon - \frac{1}{2} m\right);$$

$$\therefore \text{ Annual Precession} = 32155'' \left(\epsilon - \frac{1}{2} m\right).$$

But the Precession $= 50''.1$ by observation;

$$\therefore \epsilon - \frac{1}{2} m = 50.1 \div 32155 = 0.0015581;$$

$$\therefore \epsilon = 0.0015581 + 0.0017271 = \frac{1}{304}.$$

This agrees closely with the result from the Moon's motion (Art 145). They are both smaller than the result deduced by the fluid theory (Art. 118), and still smaller than that from pendulum experiments (Art. 143). See the remarks in the last paragraph but one of Art. 142.

147. The argument for the particular law of density deduced from these calculations is not very strong. For, as the strata are nearly spherical, almost any law might lead to right

results by choosing the constants involved in the law rightly; especially as those results are the resultant effects of the whole mass, and not the effect of the parts taken separately.

148. The ellipticity obtained by the four methods hitherto used, viz. the fluid theory, pendulum experiments, the Moon's motion in latitude, and the precession of the equinoxes, is

$$\epsilon = \frac{1}{293}, \ \frac{1}{288}, \ \frac{1}{305}, \ \frac{1}{304}.$$

We now proceed to the geodetic method and shall see that it coincides almost exactly with the first of these.

CHAPTER III.

THE FIGURE OF THE EARTH, DETERMINED BY GEODETIC OPERATIONS.

149. We have shown that if the Earth be considered a fluid mass the form of the surface will be an oblate spheroid of small ellipticity, its axis coinciding with the axis of revolution, and the surface being everywhere at right angles to the direction of gravity; and further, that upon assuming that the density of the strata varies according to a certain probable law, the ellipticity is somewhat greater than $\frac{1}{300}$, viz. $\frac{1}{293}$.

We propose to submit this to the test of measurement, by enquiring whether an ellipse can be found with its minor axis coinciding with the axis of the Earth and cutting the plumb-line at stations along it at right angles; and whether the ellipticity of that ellipse is about the amount above stated.

The method of doing this is as follows. A base-line, about 5 or 6 miles in length, is measured with extreme accuracy, near the meridian, the curvature of which we are to find. By a series of triangles this base is connected with a number of stations in succession lying near the meridian, the angles and sides of which are calculated or observed, as the case may be. Thus a connexion is established between the original base and a second base at the termination of the chain of triangles, and the length of this second base obtained by calculation. It is then measured, as the first was, and by a comparison of the calculated and measured results the correctness or not of the operations is tested. This having been satisfactorily performed, the projections of the sides of the triangles on the meridian are found, and their sum gives the length of the

meridian arc between its two extremities. The latitudes of these extremities are then observed with great care, and from these data the form of the ellipse, of which the arc is a part, is found by the principles of conic sections, as we shall now show.

It is obvious that the actual surface of the earth is of a very irregular form, being diversified by mountains and valleys. In our investigation, at any rate in the first instance, these are not taken account of; the whole is supposed to be levelled down and all the measures which are taken are reduced down to the sea-level, the sea being supposed to have the spheroidal form, since it is a free surface. The sea-level at any place means the level at which sea-water would stand if let in from the sea by a canal.

§ 1. *The determination of the Mean Figure of the Earth, assuming it to be spheroidal.*

PROP. *To find the length of an arc of meridian in terms of the amplitude, the semi-axis major, the ellipticity (the ellipticity being small), and the middle latitude.*

150. Let l and l' be the latitudes of the extremities of the arc, m the mean of these or the middle latitude; λ the amplitude of the arc or the difference between the latitudes; a, b, and ϵ the semi-axes and ellipticity; s the length of the arc, r the radius vector, and θ the angle r makes with the major axis. Then

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}, \quad \tan l=\frac{a^2}{b^2}\tan\theta;$$

$$\therefore \frac{1}{r^2}=\frac{a^2\cos^2 l+b^2\sin^2 l}{a^4\cos^2 l+b^4\sin^2 l}, \text{ putting } b=a(1-\epsilon),$$

$$r=a(1-\epsilon\sin^2 l), \text{ neglecting } \epsilon^2 \ldots$$

$$\frac{dr}{dl}=-2a\epsilon\sin l\cos l, \quad \frac{d\theta}{dl}=1-2\epsilon+4\epsilon\sin^2 l;$$

$$\therefore \frac{ds}{dl} = \sqrt{r^2 \frac{d\theta^2}{dl^2} + \frac{dr^2}{dl^2}} = a\,(1 - 2\epsilon + 3\epsilon \sin^2 l)$$

$$= a \left(1 - \frac{1}{2}\epsilon - \frac{3}{2}\epsilon \cos 2l\right);$$

$$\therefore s = a \left\{\left(1 - \frac{1}{2}\epsilon\right)(l - l') - \frac{3}{4}\epsilon\,(\sin 2l - \sin 2l')\right\}$$

$$= \frac{1}{2}(a + b)\,\lambda - \frac{3}{2}(a - b) \sin \lambda \cos 2m.$$

151. If λ be small, not exceeding 12°, we may put $\sin \lambda = \lambda$ in this formula; then

$$\frac{s}{\lambda} = \frac{a+b}{2} - 3\frac{a-b}{2} \cos 2m.$$

152. The value of λ in terms of s including the square of the ellipticity is given by the following formula, which may easily be deduced by the process in Art. 150, viz.:

$$\lambda = \frac{s}{a}\left[1 + \epsilon\left(\frac{1}{2} + \frac{3}{2}\frac{\sin\lambda}{\lambda}\cos 2m\right) + \epsilon^2\left\{\left(\frac{1}{2} + \frac{1}{3}\frac{\sin\lambda}{\lambda}\cos 2m\right)^2 \right.\right.$$
$$\left.\left. + \frac{3}{16} + \frac{3}{4}\frac{\sin\lambda}{\lambda}\cos 2m - \frac{15}{32}\frac{\sin\lambda}{\lambda}\cos\lambda\cos 4m\right\}\right].$$

153. Let S be the length of an arc of longitude in latitude l, L the longitudinal amplitude of the arc. Then the radius of the circle of longitude

$$= r\cos\theta = a\cos l\,(1 + \epsilon\sin^2 l) = \cos l\,\{a + (a - b)\sin^2 l\};$$
$$\therefore S = L\cos l\,\{a + (a - b)\sin^2 l\}.$$

PROP. *To obtain formulæ for finding the semi-axes and ellipticity, when the lengths, amplitudes, and middle latitudes of two small arcs are known; and to ascertain what arcs are adapted to give the best results.*

154. Let $s\lambda m$, $s'\lambda' m'$ be the lengths, amplitudes, and middle latitudes;

$$\therefore \frac{s}{\lambda} = \frac{a+b}{2} - 3\frac{a-b}{2}\cos 2m, \quad \frac{s'}{\lambda'} = \frac{a+b}{2} - 3\frac{a-b}{2}\cos 2m';$$

$$\therefore \frac{a-b}{2} = \frac{1}{3}\frac{\frac{s}{\lambda} - \frac{s'}{\lambda'}}{\cos 2m' - \cos 2m}, \quad \frac{a+b}{2} = \frac{\frac{s}{\lambda}\cos 2m' - \frac{s'}{\lambda'}\cos 2m}{\cos 2m' - \cos 2m},$$

by which a and b and therefore ϵ are found.

The effect on the axes of any error in the amplitudes will be found by differentiating the above formulæ. In the denominators of the resulting expressions the quantity

$$\cos 2m - \cos 2m'$$

will appear. The errors in the axes consequent on errors in the observed amplitudes will, therefore, be least when this quantity is a maximum. Suppose one arc is chosen in the southern half of the quadrant, $\cos 2m$ is positive; then

$$2m' = 180^\circ \text{ or } m' = 90^\circ$$

will give the best result. Suppose one arc is in the northern half, $\cos 2m$ is negative; then $2m' = 0$ will give the best result. Hence the nearer one arc is to the pole and the other to the equator, the less will errors in the data affect the calculated form of the ellipse. This will be illustrated in the following examples. The data are taken from the Volume of the *British Ordnance Survey*, pp. 743, 757.

155. Ex. 1. Compare the two parts of the English Arc, from Saxaford (60° 49′ 39″) to Clifton (53° 27′ 30″), measuring 2692754 feet, and from Clifton to Southampton (50° 54′ 47″), measuring 928774 feet.

$$\lambda = 7^\circ\, 22'\, 9'' = 26529'', \quad \lambda' = 2^\circ\, 32'\, 43'' = 9163'',$$

$$2m = 114^\circ\, 17'\, 9'', \quad 2m' = 104^\circ\, 22'\, 17'',$$

$$\therefore \frac{1}{2}(a-b) = 59419, \quad \frac{1}{2}(a+b) = 20863630.$$

$$a = 20923049, \quad b = 20804211, \quad \epsilon = \frac{1}{176} = \frac{1+0{\cdot}67}{295}.$$

Ex. 2. Compare the two parts of the Indian Arc from Kaliana (lat. 29° 30′ 48″) to Kalianpur (24° 7′ 11″), the length being 1961138 feet, and that between Kalianpur and Damargida (18° 3′ 15″), the length being 2202905 feet.

$$\lambda = 5^\circ\, 23'\, 37'' = 19417'', \quad \lambda' = 6^\circ\, 3'\, 56'' = 21836'',$$

$$2m = 53^\circ\, 37'\, 59'', \quad 2m' = 42^\circ\, 10'\, 26'',$$

$$\therefore\ \frac{1}{2}(a-b) = 54064, \quad \frac{1}{2}(a+b) = 20929075 \text{ feet},$$

$$a = 20983139, \quad b = 20875011, \quad \epsilon = \frac{1}{194} = \frac{1+0.52}{295}.$$

Ex. 3. Compare the arc between Kalianpur and Damargida with that between Damargida and Punnæ (8° 9′ 31″), the length being 3591784 feet.

$$\lambda = 6^\circ\, 3'\, 56'' = 21836'', \quad \lambda' = 9^\circ\, 53'\, 44'' = 35624'',$$

$$2m = 42^\circ\, 10'\, 26'', \quad 2m' = 26^\circ\, 12'\, 46'';$$

$$\therefore\ \frac{1}{2}(a-b) = 26194, \quad \frac{1}{2}(a+b) = 20867130 \text{ feet},$$

$$a = 20893324, \quad b = 20840936, \quad \epsilon = \frac{1}{399} = \frac{1-0.26}{295}.$$

Ex. 4. Compare the arcs between Kaliana and Kalianpur and between Damargida and Punnæ.

$$\lambda = 5^\circ\, 23'\, 37'' = 19417'', \quad \lambda' = 9^\circ\, 53'\, 44'' = 35624'',$$

$$2m = 53^\circ\, 37'\, 59'', \quad 2m' = 26^\circ\, 12'\, 46'';$$

$$\therefore\ \frac{1}{2}(a-b) = 39867, \quad \frac{1}{2}(a+b) = 20903830,$$

$$a = 20943697, \quad b = 20863963, \quad \epsilon = \frac{1}{262} = \frac{1+0{\cdot}13}{295}.$$

Ex. 5. Compare the arcs between Damargida and Punnæ and between Clifton and Southampton.

$$\lambda = 9^{\circ}\,53'\,44'' = 35624'', \quad \lambda' = 2^{\circ}\,32'\,43'' = 9163'',$$

$$2m = 26^{\circ}\,12'\,46'', \quad 2m' = 104^{\circ}\,22'\,17'';$$

$$\therefore \frac{1}{2}(a-b) = 32208, \quad \frac{1}{2}(a+b) = 20883305,$$

$$a = 20915513, \quad b = 20851097, \quad \epsilon = \frac{1}{325} = \frac{1-0.09}{295}.$$

It will be seen in these successive examples that the ellipticity is nearer and nearer to that which, as we shall hereafter see, is deduced from geodesy, and which so very closely accords with that obtained from the fluid theory; when the arcs compared are near each other the resulting ellipticity differs much from that value; but when they are more distant from each other, as in the fifth example, the result is far more accordant. This agrees with what was deduced from the formulæ in the last Article. If there were no errors in the data, viz. in the observed amplitudes and measured arcs, the results ought to come out in complete accordance with each other, if the figure of the Earth be truly spheroidal; for the formulæ are sufficiently exact for this purpose.

PROP. *To explain the cause of the ellipses, determined from the several pairs of arcs, differing from each other.*

156. We have assumed, (1) that the meridian arc is an ellipse, that being the form which it would have were the Earth fluid: (2) that the plumb-line at all stations of the meridian is a normal to this ellipse. These suggest in what direction we are to look for an explanation of the discrepancies in the results.

First. It is obvious that the form of equilibrium no longer actually exists, as all the variety of hill and dale, mountain and table-land and ocean-surface, sufficiently testifies. Geology teaches the same more generally and philosophically. Extensive portions now dry land were once at the bottom of the ocean, receiving the fossil deposits and burying them in the detritus of rocks, which time wore down, to become, as they are now, the records of their own history. Changes of level must therefore have taken place on a large scale.

Landmarks in Scandinavia, the temple of Serapis at Puzzuoli, the ancient and recent coral-reefs in the Pacific, all testify that these changes of level are still slowly going on. It has been suggested, with great probability, that it is caused by the expansion and contraction of vast portions of rock in the interior of the Earth arising from variations in temperature produced by chemical changes. Whatever the cause, the fact is certain. The Earth's form can no longer be a form of fluid-equilibrium, although the average form may be so.

Secondly. The plumb-line may not in all cases be perpendicular to the mean ellipse. Local attraction is sufficient to produce material errors in the vertical, and therefore in the amplitudes determined by meridian zenith distances of stars. For instance (Art. 81, Ex. 2), an error as great as 5″ was discovered at Takal K'hera in Central India by Sir G. Everest, arising from the attraction of a distant table-land. Sir Henry James has shown that a deflection of about the same amount occurs at Arthur's Seat, Edinburgh (*Phil. Trans.* 1857). We have mentioned that the attraction of the Himalaya Mountains produces a deflection amounting to as much as 28″ at the northern extremity of the Great Indian Arc (Art. 89, Ex. 1). We have calculated elsewhere (see Art. 89, Ex. 2, and *Phil. Trans.* for 1859) that the deficiency of matter in the vast ocean south of India causes such deflections as 6″, 9″, 10″.5, 19″.7 at various stations: and (Art. 94) we have shown that it is not improbable that extensive but slight variations of density prevail in the interior of the Earth, the causes of which are not visible to us as mountain masses and vast oceans are, sufficient to produce errors in the plumb-line quite as great as and even greater than most of those already enumerated. These seem abundantly to account for the variety in the calculated semi-axes and ellipticities in the last Article, derived as they are from uncorrected observations.

157. Mr Airy has entered very thoroughly into a comparison (see *Figure of the Earth*, *Encyc. Metrop.*) of the various arcs measured in different parts of the world. He has used them according to their importance and value, as determined by the circumstances under which they were measured and observed.

158. The late M. Bessel devised a method by which the results of all the surveys in different parts of the world might be brought to bear simultaneously upon the problem. This method is followed by Captain A. Clarke, R.E. in his Chapter on the figure of the earth at the end of the *British Ordnance Survey* Volume. The arcs which he uses in his calculation for determining the mean figure of the earth are eight in number; viz. the Anglo-Gallic, Russian, Indian II (or Great Arc), Indian I, Prussian, Peruvian, Hanoverian, and Danish arcs. These consist of fifty-eight subordinate divisions, the lengths of which have been measured and the latitudes of their extremities observed. The method which Bessel invented was this: corrections, expressed in algebraical terms, are applied to the latitudes of the several stations dividing the arcs into their subordinate parts, such as to make their measured lengths exactly fit an ellipse. The values of the axes of this ellipse are then so determined as to make the sum of the squares of these corrections a minimum: *that* is the ellipse which most nearly represents the observations and measures, and is therefore taken to be the mean ellipse.

PROP. *To obtain a formula for correcting the amplitude of an arc, so as to make its measured length accord with a given ellipse.*

159. The length of an arc is, by Art. 150,

$$s = \frac{1}{2}(a+b)\lambda - \frac{3}{2}(a-b)\sin\lambda\cos 2m.$$

Suppose now that x x' are small corrections which must be applied to the observed latitudes to make the measured arc fit the ellipse of which a and b are the semi-axes; λ and m, being obtained from observation, will not, when substituted in the above formula, give the measured value of s; but

$$\lambda + x' - x \text{ and } 2m + x' + x$$

must be substituted instead of them. Hence, omitting very small quantities,

$$2s=(a+b)\lambda-3(a-b)\sin\lambda\cos 2m$$
$$+(x'-x)\{a+b-3(a-b)\cos\lambda\cos 2m\};$$

$$\therefore x'-x=\left(\frac{2s}{a+b}-\lambda+3\frac{a-b}{a+b}\sin\lambda\cos 2m\right)\left(1+3\frac{a-b}{a+b}\cos\lambda\cos 2m\right).$$

Now the mean radius of the earth is known not to differ much from 20890000 feet, and the ellipticity from $1\div 300$. It is therefore convenient to put a and b under the form

$$\frac{a+b}{2}=\left(1-\frac{u}{10000}\right)20890000,$$

$$\frac{a-b}{2}=\frac{1}{600}\left(1-\frac{u}{10000}+\frac{v}{50}\right)20890000;$$

$$\text{and } \therefore\ \epsilon=\frac{1}{300}\left(1+\frac{v}{50}\right);$$

where the squares of u and v may be neglected. When these are substituted in the formula it may be put in the following form,

$$x'=m+\alpha u+\beta v+x,$$

where m, α, β are functions of the observed latitudes and the measured length and other numerical quantities only. Their values have been calculated in the *British Ordnance Survey* Volume for the 85 divisions of the 8 arcs mentioned in Art. 158.

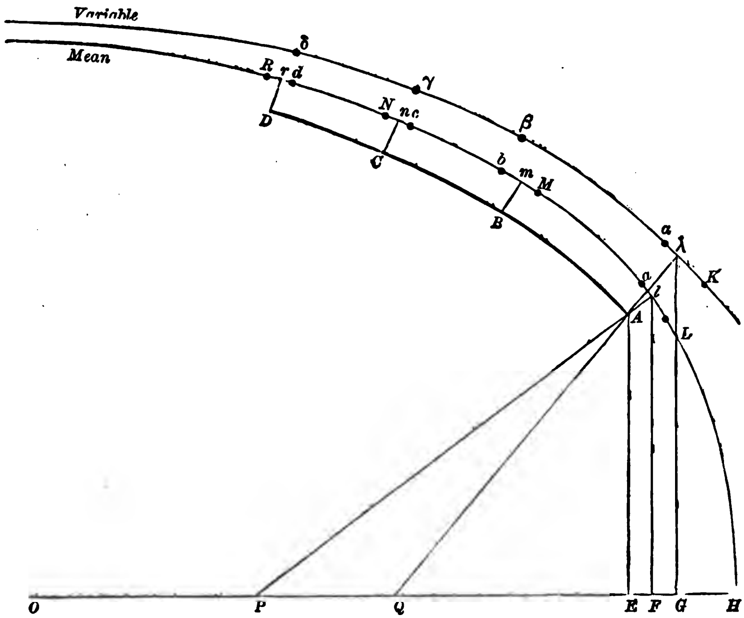
160. In pursuing the process described in Art. 158, the ten quantities u, v, x_1, $x_2 \ldots x_8$ are *all considered as variables* in Bessel's method, to be determined so as to make the sum of the squares of the corrections a minimum. The result is, that $u=-0.3856$, $v=1.0620$; and these make

$$a=20926348,\ b=20855233,\ \epsilon=\frac{1}{294}.$$

This process, we think, is not correct: for although $x_1 \ldots x_8$ are unknown quantities, yet they are not variables independent of u and v. This will appear in the course of the following Proposition.

PROP. *To obtain equations to calculate the axes of the ellipse which best represents a given arc.*

161. In the accompanying diagram, let $ABCD$ be the level-curve or arc, A, B, C, D being the stations at which the latitudes are observed, the lengths of the intermediate curves being measured by the Trigonometrical Survey. Suppose $\alpha\delta$ is an ellipse of small ellipticity and variable axes, not differing much from the arc AD in form and position. Let a and b be the semi-axes of this ellipse, and ϵ its ellipticity.



Suppose $\alpha\beta\gamma\delta$ is the portion of this ellipse which represents the arc $ABCD$, the mutual distances of α, β, γ, δ being the same as of A, B, C, D; and let K be the point at which the normal is parallel to the plumb-line at A, and let $K\alpha = x$; x, measured in parts of a terrestrial degree, is evidently the small angle which must be added to the observed latitude of A to obtain the latitude of α. The corresponding angles

which must be added to the observed latitudes of B, C, D are (see Art. 159)

$$m' + \alpha' u + \beta' v + x, \quad m'' + \alpha'' u + \beta'' v + x, \ldots$$

where m', α', β',...are numerical quantities depending upon observed latitudes and measured lengths of arcs. Those values of u and v which will make the sum of the squares of these corrections a minimum are those which give the ellipse which most nearly coincides with the arc in question. This is called the Mean Ellipse. Let U and V be the values of u and v which correspond to this ellipse; and suppose LR is this mean ellipse.

162. Before we apply the principle of least squares it is necessary to determine the form of x, which is as yet an arbitrary quantity. With a view to do this we must consider what part of the mean ellipse properly represents the arc $ABCD$. Suppose a, b, c, d are the positions which α, β, γ, δ assume when the variable ellipse coincides with the mean ellipse, these points being at the same mutual distances measured along the ellipse as the stations A, B, C, D are from each other. The correction x must be so determined as to make $abcd$ suitably represent the position of the geodetic arc $ABCD$ when referred to the mean ellipse.

Draw Al, Bm, Cn, Dr normals to the mean ellipse; then, if the points l, m, n, r are at the same distance from each other as A, B, C, D are, the arc $lmnr$ will properly represent $ABCD$, as it will be simply its projection upon the mean ellipse. This will not, however, be generally the case. The points a, b, c, d must therefore be as near to l, m, n, r as possible; that is, the sum of the squares of the distances of a, b, c, d from l, m, n, r must be a minimum. As the relative distances of a, b, c, d are fixed, there is only one variable quantity, viz. the distance of a from l; on this the other distances depend. The condition that the sum of the squares should be a minimum leads to this other condition, that the algebraical sum of the distances of a, b, c, d from l, m, n, r should equal zero. This will give the value of la; it is a constant but unknown quantity depending upon the form of the mean ellipse; let it for the present be called z. The arc

abcd thus defined is the portion of the mean ellipse which represents the measured arc *ABCD*.

We will now calculate the form of z. Let L, M, N, R be the points on the mean ellipse at which the normals are parallel to the plumb-line at A, B, C, D. Then the arcs Ll, Mm, nN, rR represent the local deflections at A, B, C, D*, as they are the arcs which at the centre of the earth measure the angles between the plumb-line and the normal to the mean ellipse at the station. Let them be called t, t', t'', t''', the deflection being reckoned positive when it is to the north. Then $la = La - t$. Also La being the increment added to the observed latitude of A which gives the latitude of a, then Mb, the corresponding increment to the observed latitude of B to obtain the latitude of b, will, by the formulæ of Art. 159, be

$$La + m' + \alpha' U + \beta' V,$$

$$\therefore\ mb = Mb - Mm = La + m' + \alpha' U + \beta' V - t'.$$

In the same way the distances of c and d from n and r are

$$La + m'' + \alpha'' U + \beta'' V - t'',$$

$$La + m''' + \alpha''' U + \beta''' V - t'''.$$

Let (m) be a symbol which represents the sum of all the ms, and so of the other quantities. Then the condition deduced in the last paragraph, that the algebraical sum of the distances of all the points a, b, c, d from l, m, n, r is to be zero, gives

$$0 = 4La + (m) + (\alpha)\,U + (\beta)\,V - (t);$$

or generally, if i represents the number of stations on the arc,

$$0 = i\,.\,La + (m) + (\alpha)\,U + (\beta)\,V - (t).$$

$$\text{But } La = Ll + la = t + z;$$

$$\therefore z = \frac{(t)}{i} - t - \frac{(m) + (\alpha)\,U + (\beta)\,V}{i}.$$

* Of course in the diagram small quantities are enormously exaggerated to make them visible to the eye.

Having found z, we proceed to find the remaining part of x. Draw $A\lambda$, the normal from A on the variable ellipse; the position of the arc $\alpha\beta\gamma\delta$ being so chosen as to make $\lambda\alpha =$ constant z, the point α will fall upon a when the variable ellipse coincides with the mean ellipse, and also λ with l. Then x, the correction to be added to the observed latitude of A to obtain the point α in the variable ellipse, that is, the arc $K\alpha$,

$$= K\lambda + \lambda\alpha = K\lambda + z,$$

$K\lambda$ has, then, to be found.

If the earth had its mean form, the plumb-line at A would hang in the normal lA to the mean ellipse. Hence the angle which the plumb-line at A makes with lA is the change in the observed latitude of A arising from local attraction, and therefore equals t. This, then, is the angle between the normal at K and the line lA. Add to this the angle $lA\lambda$, and we have the whole angle measured by the arc $K\lambda$.

This angle $lA\lambda$ can be found by conic sections as follows:—Draw AE, lF, λG perpendicular to OH, the semi-axis major, and produce the normals lA, λA to P and Q. Let ϵ and ϵ' be the ellipticities of the mean and variable ellipses. Then by conic sections $OP = 2\epsilon . OF$, $OQ = 2\epsilon' . OG$. Hence, neglecting small quantities of the second order,

$$\tan lA\lambda = \tan (AQH - APH) = \frac{\cot APH - \cot AQH}{1 + \cot APH . \cot AQH}$$

$$= \frac{(PE - QE)\, AE}{AE^2 + PE . QE} = \frac{(OQ - OP)\, AE}{AE^2 + OE^2}$$

$$= 2\,(\epsilon' - \epsilon) \sin l \cos l = (\epsilon' - \epsilon) \sin 2l,$$

l being the observed latitude of A;

$$\therefore\ lA\lambda = (\epsilon' - \epsilon) \sin 2l \frac{1''}{\sin 1''},$$

or, by formulæ of Art. 159,

$$= \frac{\sin 2l}{15000 \sin 1''} (v - V) = 13''.75 \sin 2l\, (v - V) = n\, (v - V);$$

$$\therefore\ K\lambda = t + n\, (v - V).$$

Having thus found the several parts of x, we have finally,

$$x = t + n(v - V) + z$$
$$= n(v - V) + \frac{(t)}{i} - \frac{(m) + (\alpha)U + (\beta)V}{i},$$
$$\text{also } \frac{dx}{du} = 0, \quad \frac{dx}{dv} = n.$$

163. We are now ready to apply the principle of least squares to discover the form of the mean ellipse. The sum of the squares of the corrections of latitudes which is to be made a minimum is

$$x^2 + (m' + \alpha' u + \beta' v + x)^2 + (m'' + \alpha'' u + \beta'' v + x)^2 + \dots$$

Differentiating first with respect to u and next to v, and then putting U and V for u and v, we have

$$0 = (m' + \alpha' U + \beta' V + x)\,\alpha' + (m'' + \alpha'' U + \beta'' V + x)\,\alpha'' + \dots$$
$$0 = xn + (m' + \alpha' U + \beta' V + x)(\beta' + n)$$
$$+ (m'' + \alpha'' U + \beta'' V + x)(\beta'' + n) + \dots$$

or, introducing symbols,

$$0 = (m\alpha) + (\alpha^2)U + (\alpha\beta)V + x(\alpha),$$
$$0 = \{ni + (\beta)\}\,x + (m\beta) + (\alpha\beta)U + (\beta^2)V$$
$$+ n(m) + n(\alpha)U + n(\beta)V;$$

or if we substitute for x, first putting V for v,

$$0 = (m\alpha) - \frac{(m)(\alpha)}{i} + \left\{(\alpha^2) - \frac{(\alpha)^2}{i}\right\} U$$
$$+ \left\{(\alpha\beta) - \frac{(\alpha)(\beta)}{i}\right\} V + \frac{(\alpha)}{i}(t),$$

$$0 = (m\beta) - \frac{(m)(\beta)}{i} + \left\{(\alpha\beta) - \frac{(\alpha)(\beta)}{i}\right\} U$$
$$+ \left\{(\beta^2) - \frac{(\beta)^2}{i}\right\} V + \left\{n + \frac{(\beta)}{i}\right\}(t).$$

These agree with the equations obtained by Bessel's method (see Captain Clarke's calculation, *Ordnance Survey* Volume, p. 738), with the exception of the terms which depend upon (t), or the sum of the local deflections at all the stations of the arc.

It will be seen, as might have been anticipated, that taking into consideration the effect of local attraction introduces an element of uncertainty, as there is no known method of finding what that sum is.

164. COR. 1. The value of x for the Mean Ellipse is

$$\frac{(t)}{i} - \frac{(m) + (\alpha) U + (\beta) V}{i},$$

from this and the values of m, α, β for the several stations, the corrections of latitude to make the observed latitudes suit the mean ellipse can be at once found when the mean axes and sum of local attractions are known, by the formula

$$m + \alpha U + \beta V + x.$$

COR. 2. From the reasoning in Art. 161—2 we can deduce the local attraction of any particular station of an arc when we know the sum of the local attractions of all the stations. Thus

$$\begin{aligned} t'' &= m'' + \alpha'' U + \beta'' V + x \\ &= m'' + \alpha'' U + \beta'' V \\ &\qquad - \frac{(m) + (\alpha) U + (\beta) V}{i} + \frac{(t)}{i}, \end{aligned}$$

this is the angle through which the plumb-line is drawn north, and therefore by which the latitude is diminished in consequence of local attraction. Hence it is the correction which must be applied to the latitude to make the latitude suit the mean ellipse.

165. If two or more arcs are combined to find the axes, then (Σ being a symbol meaning that the sum of all such

quantities as are placed after it is to be taken, passing from arc to arc), the equations will be

$$0 = \Sigma\left\{(m\alpha) - \frac{(m)\,(\alpha)}{i}\right\} + \Sigma\left\{(\alpha^2) - \frac{(\alpha)^2}{i}\right\}.\,U$$

$$+ \Sigma\left\{(\alpha\beta) - \frac{(\alpha)\,(\beta)}{i}\right\}.\,V + \Sigma\,\frac{(\alpha)}{i}\,(t),$$

$$0 = \Sigma\left\{(m\beta) - \frac{(m)\,(\beta)}{i}\right\} + \Sigma\left\{(\alpha\beta) - \frac{(\alpha)\,(\beta)}{i}\right\}.\,U$$

$$+ \Sigma\left\{(\beta^2) - \frac{(\beta^2)}{i}\right\}.\,V + \Sigma\left\{n + \frac{(\beta)}{i}\right\}(t).$$

The axes, deduced from several independent arcs taken together, will contain as many unknown quantities as the number of independent arcs employed.

166. The fundamental difference between the method here given and Bessel's consists in the different positions which the methods assign to the geodetic arc when referred to the mean ellipse. In the author's method the projection of the arc on the mean ellipse is taken to be the portion of the mean ellipse which represents the arc, that is, the position of *abcd* (which represents the geodetic arc *ABCD* when referred to the mean ellipse) is so determined that the algebraical sum of the departures of *a*, *b*, *c*, *d* from the normals at *l*, *m*, *n*, *r* is zero. In Bessel's method, *L*, *M*, *N*, *R* being the points on the mean ellipse at which the normals are parallel to the plumb-lines at *A*, *B*, *C*, *D*, the position of *abcd* is so determined that the algebraical sum of the departures of *a*, *b*, *c*, *d* from *L*, *M*, *N*, *R* is zero. But the positions of *L*, *M*, *N*, *R* are affected by the unknown local attraction at *A*, *B*, *C*, *D*. Hence by Bessel's process the arc *abcd* may lie to the right or left of the projection of the arc *ABCD* on the mean ellipse. It will be at once seen that it receives this side-shift in consequence of its position, as determined by Bessel's method, depending upon unknown local attraction which in his method is not provided for. This misplacement of the geodetic arc influences the form of the final equations for determining the mean axes.

167. The only way which presents itself of overcoming the difficulty, mentioned at the end of Art. 163, in obtaining the mean axes, arising from unknown local attraction, is the following: Find the axes of the three long arcs (the Anglo-Gallic, the Russian, and the Indian), separately, by means of the equations, in terms of the three unknown sums (t_1), (t_2), (t_3); make these axes the same in the three ellipses, which process will give four equations connecting these three unknowns, and then by the method of least squares find the values of (t_1), (t_2), (t_3)—which best suit the equations—which values, when they are thus found, will determine the numerical values of the mean axes. This we proceed to do. We would observe, that making the axes of the three arcs equal to each other amounts to assuming that the mean form of the earth is an ellipsoid of revolution, an assumption which has been made by every other investigator in this subject.

PROP. *To apply the equations for the mean axes to the three long arcs, the Anglo-Gallic, Russian, and Indian separately, and by their comparison to obtain a Mean Figure of the Earth from them.*

168. The data given in the following Table are gathered from the *British Ordnance Survey* Volume, pp. 766—768, and the *Monthly Notices of the Royal Astronomical Society*, Vol. XIX. p. 35, and the values of n are calculated from the formula $n = 13.75 \sin 2l$ (Art. 162). The meaning of brackets enclosing an algebraical symbol, thus (t_1), has been already explained to be, that the sum of all quantities like that within the brackets is to be taken for all the stations of the arc under consideration. The meaning of brackets when a logarithm is enclosed within them is this, that the number is to be taken of which the number within the brackets is the logarithm; thus $58.6109 = (1.7679784)$.

I. *The Anglo-Gallic Arc.*

169. The equations of Art. 163 for finding the values of U_1 and V_1 become, when the numbers are substituted from the Table,

Arcs.	$(m\alpha)$.	$(m\beta)$.	(α^2).	$(\alpha\beta)$.	(β^2)	i.	n.
Anglo-Gallic.	+118.9207	− 45.3653	+155.0671	− 35.5339	+11.1906	34	13.5500
Russian	+386.3623	−126.4488	+335.5318	−112.1421	+39.9745	13	13.7491
Indian	− 12.7516	− 10.4936	+ 46.5750	+ 36.6984	+29.3612	8	8.1031
	$-\frac{(m)(\alpha)}{i}$.	$-\frac{(m)(\beta)}{i}$.	$-\frac{(\alpha)^2}{i}$.	$-\frac{(\alpha)(\beta)}{i}$.	$-\frac{(\beta)^2}{i}$.	$\frac{(\alpha)}{i}$.	$\frac{(\beta)}{i}$.
Anglo-Gallic.	− 60.3098	+ 24.3352	− 27.7474	+ 11.1962	− 4.5177	+0.9034	− 0.3645
Russian	−291.4511	+ 85.4950	−228.4246	+ 67.0066	−19.6559	+4.1918	− 1.2296
Indian	+ 0.3492	+ 0.5757	− 0.3135	− 0.5168	− 0.8521	−0.1979	− 0.3264
The sums of the above quantities.							
Anglo-Gallic.	+ 58.6109	− 21.0301	+127.3197	− 24.3377	+ 6.6729	+0.9034	+13.1855
Russian	+ 94.9112	− 40.9538	+107.1072	− 45.1355	+20.3186	+4.1918	+12.5195
Indian	− 12.4024	− 9.9179	+ 46.2615	+ 36.1816	+28.5091	−0.1979	+ 7.7767

$$58.6109 + 127.3197\,U_1 - 24.3377\,V_1 + \ 0.9034\,(t_1) = 0,$$
$$-21.0301 - \ 24.3377\,U_1 + \ 6.6729\,V_1 + 13.1855\,(t_1) = 0,$$

or

$$(1.7679784) + (2.1048957)U_1 - (1.3862795)V_1$$
$$+ \bar{1}.9558801)\,(t_1) = 0,$$
$$-(1.3228414) - (1.3862795)U_1 + (0.8243146)\,V_1$$
$$+ (1.1200966)\,(t_1) = 0.$$

Eliminating V_1 by multiplying crosswise by its coefficient and adding,

$$\left.\begin{matrix}(2.5922930)+(2.9292103)\\ -(2.7091209)-(2.7725590)\end{matrix}\right\} U_1 + \left.\begin{matrix}(0.7801947)\\ +(2.5063761)\end{matrix}\right\}(t_1)=0,$$

or

$$\begin{array}{l}\left.\begin{matrix}391.10+849.59\\ -511.82-592.32\end{matrix}\right\} U_1 + \left.\begin{matrix}6.03\\ +320.90\end{matrix}\right\}(t_1)\\ \hline -120.72+257.27\,U_1 \quad +326.93\,(t_1)=0,\end{array}$$

or

$$-(2.0817792)+(2.4103891)\,U_1+(2.5144548)\,(t_1)=0\,;$$

$$\therefore\ U_1=(\bar{1}.6713901)-(0.1040657)\,(t_1).$$

From the first of the equations in V_1 we have

$$\begin{aligned}V_1&=(0.3816989)+(0.7186162)\,U_1+(\bar{2}.5696006)\,(t_1)\\ &=\left.\begin{matrix}(0.3816989)-(0.8226819)\\ +(0.3900063)+(\bar{2}.5696006)\end{matrix}\right\}(t_1)=\left.\begin{matrix}2.4082-6.6479\\ +2.4547+0.0371\end{matrix}\right\}(t_1)\\ &=4.8629-6.6108\,(t_1)=(0.6868953)-(0.8202540)\,(t_1).\end{aligned}$$

Hence, by the formulæ of Art. 159,

$$\begin{aligned}\frac{a_1+b_1}{2}&=20890000-2089\,U_1\,;\qquad 2089=(3.3199384)\,;\\ &=20890000-(2.9913285)+(3.4240041)\,(t_1)\\ &=20890000-980.2+2654.6\,(t_1)=20889020+2654.6\,(t_1)\\ &=(7.3199180)+(3.4240041)\,(t_1),\end{aligned}$$

$$\frac{a_1-b_1}{2}=\frac{1}{600}\left\{\frac{a_1+b_1}{2}+417800\,V_1\right\};\qquad \begin{matrix}4178=(3.6209684)\\ 600=(2.7781513)\end{matrix}$$

$$\begin{aligned}&=\frac{1}{600}\left\{\begin{matrix}\left.\begin{matrix}(7.3199180)+(3.4240041)\\ +(6.3078637)-(6.4412224)\end{matrix}\right\}(t_1)\end{matrix}\right\}\\ &=\left.\begin{matrix}(4.5417667)+(3.6458528)\\ +(3.5297124)-(3.6630711)\end{matrix}\right\}(t_1)\\ &=\left.\begin{matrix}34815+4.4\\ 3386-4603.3\end{matrix}\right\}(t_1)=38201-4599\,(t_1)\,;\end{aligned}$$

$$\therefore\ a_1=20927221-1944\,(t_1),$$
$$b_1=20850819+7254\,(t_1).$$

II. *The Russian Arc.*

170. Following a similar process, we have

$$94.9112+107.1072\,U_2-45.1355\,V_2+4.1918\,(t_2)=0,$$
$$-40.9538-45.1355\,U_2+20.3186\,V_2+12.5195\,(t_2)=0,$$

or

$$(1.9773174)+(2.0298187)\,U_2-(1.6545183)\,V_2 + (0.6224006)\,(t_2)=0,$$
$$-(1.6122942)-(1.6545183)\,U_2+(1.3078938)\,V_2 + (1.0975870)\,(t_2)=0,$$

$$\left.\begin{array}{r}(3.2852112)+(3.3377125)\\-(3.2668125)-(3.3090366)\end{array}\right\}U_2+\left.\begin{array}{r}(1.9302944)\\+(2.7521053)\end{array}\right\}(t_2)=0,$$

or

$$\left.\begin{array}{r}1928.46+2176.27\\-1848.47-2037.21\end{array}\right\}U_2+\left.\begin{array}{r}85.17\\+565.07\end{array}\right\}(t_2)$$
$$\overline{79.99+139.06\;U_2+650.24\;(t_2)=0,}$$

or

$$(1.9030357)+(2.1432022)\,U_2+(2.8130737)\,(t_2)=0\,;$$
$$\therefore\; U_2=-(\bar{1}.7598335)-(0.6698715)\,(t_2).$$

Also

$$\begin{aligned}V_2&=(0.3227991)+(0.3753004)\,U_2+(2.9678823)\,(t_2)\\&=\left.\begin{array}{r}(0.8227991)-(1.0451719)\\-(0.1351339)+(\bar{2}.9678823)\end{array}\right\}(t_2)=\begin{array}{r}2.1028-11.0961\\-1.3650+0.0929\end{array}\\&=0.7378-11.0032\,(t_2)=(\bar{1}.8679387)-(1.0415190)\,(t_2)\,;\end{aligned}$$

$$\therefore\;\frac{a_2+b_2}{2}=20890000-2089\,U_2\,;\quad 2089=(3\cdot3199384)\,;$$
$$\begin{aligned}&=20890000+(3.0797719)+(3.9898099)\,(t_2)\\&=20890000+1201.6+9768.1\,(t_2)=20891202+9768\,(t_2)\\&=(7.3199634)+(3.9898099)\,(t_2)\,;\end{aligned}$$

$$\frac{a_2+b_2}{2}=\frac{1}{600}\left\{\frac{a_2+b_2}{2}+417800\,V_2\right\};\quad \begin{array}{r}4178=(3.6209684)\\ 600=(2.7781513)\end{array};$$

$$=\frac{1}{600}\left\{\begin{array}{l}(7.3199634)+(3.9898099)\\+(5.4889071)-(6.6624874)\end{array}\right\}(t_2)\Bigg\}$$

$$=\left.\begin{array}{l}(4.5418121)+(1.2116586)\\+(2.7107558)-(3.8843361)\end{array}\right\}(t_2)$$

$$=\left.\begin{array}{l}34819+\quad 16\\+\quad 514-7662\end{array}\right\}(t_2)=35333-7646\,(t_2);$$

$$\therefore\ a_2=20926535+2122\,(t_2),$$
$$b_2=20855869+17414\,(t_2).$$

III. *The Indian Arc.*

171. Pursuing the same process for this arc:

$$-12.4024+46.2615\,U_3+36.1816\,V_3-0.1979\,(t_3)=0,$$
$$-\ \ 9.9179+36.1816\,U_3+28.5091\,V_3+7.7767\,(t_3)=0,$$

or

$$-(1.0935057)+(1.6652197)\,U_3+(1.5584878)\,V_3$$
$$-(\bar{1}.2964458)\,(t_3)=0,$$

$$-(0.9964197)+(1.5584878)\,U_3+(1.4549835)\,V_3$$
$$+(0.8907953)\,(t_3)=0,$$

$$\left.\begin{array}{l}-(2.5484892)+(3.1202032)\\+(2.5549075)-(3.1169756)\end{array}\right\}U_3-\left.\begin{array}{l}(0.7514293)\\-(2.4492831)\end{array}\right\}(t_3)=0,$$

or

$$\left.\begin{array}{l}-353.58+1318.9\\+358.85-1309.1\end{array}\right\}U_3-\left.\begin{array}{r}5.64\\-281.37\end{array}\right\}(t_3)$$

$$\overline{\qquad 5.27+\quad 9.8\ U_3-287.01\ \ (t_3)=0,}$$

or

$$(0.7218106)+(0.9912261)\,U_3-(2.4578970)\,(t_3)=0;$$
$$\therefore\ U_3=(1.4666709)-(\bar{1}.7305845).$$

Also

$$
\begin{aligned}
V_3 &= (\bar{1}.5350179) - (0.1067319)\,U_3 + (\bar{3}.7379580)\,(t_3) \\
&= \left.\begin{matrix}(\bar{1}.5350179) - (1.5734028) \\ + (\bar{1}.8373164) + (\bar{3}.7379580)\end{matrix}\right\}(t_3), \\
&= \left.\begin{matrix}0.34278 - 37.446 \\ + 0.68757 + 0.005\end{matrix}\right\}(t_3) = 1.03035 - 37.441\,(t_3) \\
&= (0.129848) - (1.5733474)\,(t_3);
\end{aligned}
$$

$$
\begin{aligned}
\frac{a_3 + b_3}{2} &= 20890000 - 2089\,U_3; \quad 2089 = (3.3199384); \\
&= 20890000 + (3.0505229) - (4.7866093)\,(t_3) \\
&= 20890000 + 1123 - 61180\,(t_3) = 20891123 - 61180\,(t_3) \\
&= (7.3199618) - (4.7866093)\,(t_3);
\end{aligned}
$$

$$
\begin{aligned}
\frac{a_3 - b_3}{2} &= \frac{1}{600}\left\{\frac{a_3 + b_3}{2} + 417800\,V_3\right\}; \quad \begin{matrix}4178 = (3.6209684) \\ 600 = (2.7781513)\end{matrix}; \\
&= \frac{1}{600}\left\{\begin{matrix}(7.3199618) - (4.7866093) \\ + (5.6339532) - (7.1943158)\end{matrix}\right\}(t_3) \\
&= \left.\begin{matrix}(4.5418105) - (2.0084580) \\ + (2.8558019) - (4.4161645)\end{matrix}\right\}(t_3) \\
&= \left.\begin{matrix}34818.5 - 102 \\ 717.5 - 26071\end{matrix}\right\}(t_3) = 35536 - 26173\,(t_3);
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ a_3 &= 20926659 - 87353\,(t_3), \\
b_3 &= 20855587 - 35007\,(t_3).
\end{aligned}
$$

172. We have now to find, if possible, values of the local attractions which will make these three ellipses, representing the Anglo-Gallic, Russian, and Indian arcs, the same—that is, $a_1 = a_2 = a_3$, and $b_1 = b_2 = b_3$. These lead to the following equations:

$$
\begin{aligned}
1944\,(t_1) + 2122\,(t_2) - 686 &= 0, \\
7254\,(t_1) - 17414\,(t_2) - 5050 &= 0, \\
1944\,(t_1) - 87353\,(t_3) - 562 &= 0, \\
7254\,(t_1) + 35007\,(t_3) - 4768 &= 0,
\end{aligned}
$$

or

$$\begin{aligned}
&(3.2886963)(t_1)+(3.3267454)(t_2)-(2.8363241)=0,\\
&(3.8605776)(t_1)-(4.2408985)(t_2)-(3.7032914)=0,\\
&(3.2886963)(t_1)-(4.9412778)(t_3)-(2.7497363)=0,\\
&(3.8605776)(t_1)+(4.5441549)(t_3)-(3.6783362)=0.
\end{aligned}$$

By the method of least squares we are able to obtain the most likely values of (t_1), (t_2), (t_3), which satisfy these four equations. For this end, multiply each equation by the coefficient of (t_1), and add all together; do the same for (t_2) and (t_3); and we have three equations, the solution of which gives the required quantities.

For the first equation,

$$\left.\begin{aligned}
&(6.5773926)(t_1)+(6.6154417)(t_2)-(6.1250204)\\
+\,&(7.7211552)(t_1)-(8.1014761)(t_2)-(7.5638690)\\
+\,&(6.5773926)(t_1)-(8.2299741)(t_3)-(6.0384326)\\
+\,&(7.7211552)(t_1)+(8.4047325)(t_3)-(7.5389138)
\end{aligned}\right\}=0;$$

or, rejecting 3 from the indices throughout,

$$\begin{array}{r}
\left.\begin{array}{r}3779\\52621\\3779\\52621\end{array}\right\}\\
\hline
112800
\end{array}
\begin{array}{l}
(t_1)+\left.\begin{array}{r}4125\\-126321\end{array}\right\}(t_2)-\left.\begin{array}{r}169814\\+253941\end{array}\right\}(t_3)\begin{array}{r}-\ 1334\\-36633\\-\ 1093\\-34587\end{array}\\
\hline
(t_1)-122196\ (t_2)+\ 84127\ (t_3)-73647=0,
\end{array}$$

or

$$(5.0523091)(t_1)-(5.0870570)(t_2)+(4.9249354)(t_3)$$
$$-(4.8671551)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

For the second equation,

$$\left.\begin{aligned}
&(6.6154417)(t_1)+(6.6534908)(t_2)-(6.1630695)\\
+\,&(8.1014761)(t_1)-(8.4817970)(t_2)-(7.9441899)
\end{aligned}\right\}=0;$$

or, rejecting 3 from all the indices,

$$\begin{array}{l}
\left.\begin{array}{r}4125\\126321\end{array}\right\}(t_1)+\left.\begin{array}{r}4503\\-303247\end{array}\right\}(t_2)\begin{array}{r}-\ 1456\\-87941\end{array}\\
\hline
130446\ (t_1)-298744\ (t_2)-89397=0,
\end{array}$$

or

$$(5.1154308)(t_1) - (5.4752992)(t_2) - (4.9513229) = 0\,;$$

$$\therefore\ (t_2) = (\bar{1}.6401316)(t_1) - (\bar{1}.4760237).$$

For the third equation,

$$\left.\begin{array}{r}(8.2299741)(t_1) - (9.8825556)(t_3) - (7.691014)\\ +\,(8.4047325)(t_1) + (9.0883098)(t_3) - (8.2224911)\end{array}\right\} = 0\,;$$

or, rejecting 3 from all the indices,

$$\begin{array}{rlrlr} \left.\begin{array}{r}169814\\ +253941\end{array}\right\} & (t_1) - & \left.\begin{array}{r}7630545\\ +1225490\end{array}\right\} & (t_3) - & \begin{array}{r}49092\\ -166913\end{array}\\ \hline 423755 & (t_1) - & 6405055 & (t_3) - & 216005 = 0,\end{array}$$

or

$$(5.6271149)(t_1) - (6.8065228)(t_3) - (5.3344639) = 0\,;$$

$$\therefore\ (t_3) = (\bar{2}.8205921)(t_1) - (\bar{2}.5279411).$$

Substituting for (t_2) and (t_3) in equation (a), we have

$$\left.\begin{array}{r}(5.0523091)\\ -(4.7271886)\\ +(3.7455275)\end{array}\right\} (t_1) \begin{array}{l}+\,(4.5630807) = 0,\\ -\,(3.4528765)\\ -\,(4.8671551)\end{array}$$

or

$$\begin{array}{rl} \left.\begin{array}{r}112800\\ +\quad 5566\\ -\quad 53357\end{array}\right\} & (t_1) \begin{array}{r}+\,36566\\ -\ \ 2837\\ -73647\end{array}\\ \hline 65009 & (t_1) - 39918 = 0,\end{array}$$

or

$$(4.8129735)(t_1) = (4.6011688)\,;$$

$$\therefore\ (t_1) = (\bar{1}.7881953) = 0''.610\,;$$

$$\therefore\ (t_2) = (\bar{1}.4283269) - (\bar{1}.4760237) = 0.268 - 0.299 = -0''.031,$$

and

$$(t_3) = (\bar{2}.6087874) - (\bar{2}.5279411) = 0.041 - 0.034 = 0''.007.$$

173. When these values are substituted in the values of a_1, a_2, a_3, b_1, b_2, b_3, these latter become

$$\begin{array}{rlrl} & a_1 = 20926035, & & b_1 = 20855242, \\ & a_2 = 20926469, & & b_2 = 20855329, \\ & a_3 = \underline{20926048}, & & b_3 = \underline{20855342}, \\ \text{average} = & 20926184, & & 20855304. \end{array}$$

In the semi-major-axis the greatest departure from the average is 285 feet (in the Russian arc); in the semi-minor-axis the greatest departure is 62 feet (in the Anglo-Gallic). These near coincidences of the Three Long Arcs show how well their average ellipse represents the mean ellipse. We take, therefore, the semi-axes and ellipticity of the mean ellipse to be

$$a = 20926184, \quad b = 20855304, \quad \epsilon = \frac{1}{295.2}.$$

174. We believe that the mean axes have never before been found, taking into consideration the effect of Local Attraction.

These values for the mean semi-axes differ by only 164 and 71 feet, the first in defect and the second in excess, from the values assigned in the British Ordnance Survey Volume. But this near coincidence arises simply from the fortuitous circumstance that the algebraical sum of the local attractions throughout each of the great Arcs, viz. (t_1), (t_2), (t_3), is a minute quantity. Had this not been the case, the calculation in the Chapter on the Figure of the Earth in that volume would not have been so near the truth, as unknown local attraction is not there taken into account.

175. There are no other arcs which have been measured that are of sufficient length to lead to trustworthy results. The longest of those other measured arcs is that at the Cape, and the next longest the Peruvian Arc. These are respectively $4^\circ 37'$ and $3^\circ 7'$ long: and the calculation brings out the axes as follows :—

$$a_4 = 20973241 - 31831546\,(t_4), \quad b_4 = 20865743 - 4251624\,(t_4),$$
$$a_5 = 20922449 - 1003937\,(t_5), \quad b_5 = 20854139 - 498749\,(t_5).$$

No values of (t_4) and (t_5) will make either of these pairs of values agree in any measure with the values obtained from the Three Long Arcs, and which agree so remarkably well with each other. The fact is, that the curvature of an arc of only four degrees and a half in length is far too small to indicate the form of the ellipse to which it belongs, when we bear in mind that the results are all affected by errors of observation and measurement. Of still less value, then, is the Peruvian Arc, a still shorter one; and also the Hanoverian, South-Indian, Danish, and Prussian Arcs, which measure only 2°, $1^\circ 35'$, $1^\circ 32'$, $1^\circ 30'$. These arcs can be of no use in the problem of determining the figure of the earth till they are greatly extended. They may be then used in threes, as we have done with the Anglo-Gallic, Russian, and Indian Arcs, and made to test the result already arrived at. But at present these six short arcs are of no use except for mapping the countries in which they lie. They have, however, been hitherto introduced into the general solution by Bessel's method; but as local attraction was neglected in each case, their influence on the calculation only tends to make the result the more untrustworthy.

176. It may be added that the result of the calculation in Art. 169—174 shows, that the forced hypothesis of an elliptic equator and an ellipsoidal figure is unnecessary and without any foundation. General de Schubert first suggested it in order to account for the variations in the measures of arcs, and afterwards virtually abandoned it: (see the *Monthly Notices of the Royal Astronomical Society*, Vol. XX. p. 265). He gives his method in his *Essai d'une Détermination de la veritable Figure de la Terre*, published in the *Memoirs of the Petersburgh Academy*, Vol. I. seventh series. His process is this. He finds the nearest ellipses which represent the meridians of the Russian, Indian, and French arcs, the three longest which have been measured. This he does by dividing each arc into two parts and comparing the two parts with each other or with the whole. The Russian arc, divided into two at Dorpat, latitude $58^\circ 23'$, gives for the minor-semi-axis 3261429 toises; the Indian arc, divided at Damargida, latitude $18^\circ 3'$, gives 3261547; the French arc,

divided at Carcassonne, latitude $43^\circ 13'$, gives 3260365. The first two agree very nearly. He rejects, therefore, the third, and uses the mean of the other two, giving twice the weight to the Russian that he does to the Indian: this produces 3261468 toises for the minor-semi-axis common to all meridians. With this minor-semi-axis he calculates the major-semi-axis in the Peruvian, Russian, and Indian meridians, selecting these arcs partly because of their difference of longitude. He finds the resulting semi-axes to be different, and concludes that the equator is not circular. He assumes it to be an ellipse: and finds that an ellipse with semi-axes 32726711 and 32723031 in longitude (measured from a meridian 20° west of Paris) $58^\circ 44'$ and $148^\circ 44'$ respectively will pass through those meridians at their middle points. This makes the ellipticity of the two principal meridians of the ellipsoid to be $\frac{1}{292}$ and $\frac{1}{302}$. He next computes the radii of this equatorial ellipse which correspond to the meridians of the different arcs measured in various parts of the world: these are in fact the semi-major-axes of the meridians of those arcs. With the semi-axes of the several meridians thus determined he computes the geodetic amplitudes of the several arcs and compares them with the astronomical: the following is the result:

	G. amp. – A. amp.		*G. amp. – A. amp.*
Peruvian arc	0″.077	Cape of Good Hope	–0″.442
Pennsylvanian	–6 .687	Prussian	1 .267
English (entire arc)	0 .736	Russian	–1 .289
French	–1 .607	Indian	1 .619

The Pennsylvanian, as is well known, deserves no *à priori* confidence. The other quantities are small. The Indian arc shows a difference of 2″.09 (see Art. 205), somewhat more than the difference here given. Also the measure of the French arc has been rejected without any apparent reason. So that the approximate appearance of the result must be regarded rather as accidental. Mr Airy (from whose notice of the work in the *Monthly Notices of the Royal Astronomical Society*, Vol. xx., the above remarks have been gathered) recommends that the polar semi-axis should be determined, with the other semi-axes, by a combination of the lengths of all the arcs, introducing also the latitudes of middle stations.

177. A similar calculation was afterwards made by Capt. Clarke, with Bessel's method (see *Memoirs Roy. Ast. Soc.* 1859—60, p. 25); but he neglects local attraction, as General de Schubert has done, although it is a disturbing cause which may be of more importance than any which the method of least squares is used to eliminate. In a subsequent paper indeed (as already noticed) the General points out that local attraction may greatly modify, if not altogether destroy, the discrepancies between the different meridians (see *Monthly Notices of Royal Ast. Soc.* No. 6, April 13, 1860, p. 264), a result which our calculation based upon the modification of Bessel's method fully confirms.

In the volume published in 1866 by the Ordnance Survey Department on Comparisons of Standards of Length, Captain Clarke has at p. 287 compared the results of a spheroidal and an ellipsoidal figure, and shown that the sum of the squares of the corrections, which it is necessary to apply to the latitudes to make them suit the figure, is less in the case of an ellipsoid than in that of the spheroid. In this calculation the effect of local attraction is not taken into account. His result is one which might have been foreseen. For the ellipsoid has one more disposable element than the spheroid, viz. the third axis, and therefore it naturally follows that an ellipsoid with unequal axes could be found which would fit the data of the problem more closely than a spheroid. This would be the case even if local attraction were taken into account. For even in that case there will always be some residual errors arising from observation; and it would be a singular thing, if in determining the ellipsoid which most nearly fits the data, thus affected with errors small as they may be, a figure should come out in which two of the axes are precisely equal. If a very large number of arcs in all parts of the world were measured, and, local attraction being taken into account, the result gave an ellipsoid with its two equatorial axes differing by a quantity, important when compared with the residual errors of observation, there might be some argument for an ellipsoidal figure. But short of this, the physical improbabilities of such a figure seem to be so great, that it is far more reasonable to attribute any apparent departure from the spheroid to the effects of local

attraction. There are physical reasons for supposing that the mean figure is spheroidal. But no conceivable physical cause can account for a projection or depression at each point of the equator being matched, in the mean figure, by an equal one in the opposite point of the equator and in two other points of the equator similarly related to its axes, and nowhere else. The idea of an ellipsoidal figure has sprung up solely from the mathematical side of the question, in attempting to find a continuous curve surface which lies nearest the measured geodetical arcs. But Local Attraction appears to supply a source of correction which makes a resort to so peculiar an hypothesis, as an ellipsoidal mean figure, unnecessary and untenable. This we have already shown. The following calculation shows the same in a simpler way.

PROP. *In comparing two divisions of an arc of meridian, to find the effect of a small deflection of the plumb-line at the middle station on the resulting axes.*

178. Let $\lambda+\lambda'$ be the astronomical amplitude of the whole arc, and λ and λ' the amplitudes of its two divisions. Then for determining the form of the meridian, we have, by Art. 154,

$$\frac{a-b}{2}=\frac{1}{3}\frac{\frac{s}{\lambda}-\frac{s'}{\lambda'}}{\cos 2m'-\cos 2m}, \quad \frac{a+b}{2}=\frac{\frac{s}{\lambda}\cos 2m'-\frac{s'}{\lambda'}\cos 2m}{\cos 2m'-\cos 2m}:$$

Suppose the latitude of the middle station is wrong, owing to unknown local attraction, by the quantity $\delta\lambda$: then as $\lambda+\lambda'$ is, by hypothesis, correctly determined, $\delta\lambda'=-\delta\lambda$;

$$\therefore \frac{1}{2}(\delta a-\delta b)=-\frac{1}{3}\frac{\frac{s}{\lambda^2}+\frac{s'}{\lambda'^2}}{\cos 2m'-\cos 2m}\delta\lambda.$$

By Art. 154, neglecting the ellipticity;

$$\frac{s}{\lambda}=\frac{s'}{\lambda'}=\frac{a+b}{2},$$

$$\therefore \frac{\delta a-\delta b}{a+b}=-\frac{1}{3}\frac{1}{\cos 2m'-\cos 2m}\frac{\lambda+\lambda'}{\lambda'}\frac{\delta\lambda}{\lambda}.$$

By a similar process we get

$$\frac{\delta a+\delta b}{a+b}=-\frac{\cos 2m'+\frac{\lambda}{\lambda'}\cos 2m}{\cos 2m'-\cos 2m}\,\frac{\delta\lambda}{\lambda}.$$

Ex. 1. In the Russian arc $\lambda=13^\circ 1'=46860''$ from Staro-Nekrassowka to Dorpat, $\lambda'=12^\circ 17'=44220''$ from Dorpat to Fuglenæs; and twice the middle latitudes are $103^\circ 43'$ and $129^\circ 3'$.

If $\delta\lambda=1''$ only,

$$\frac{\delta a-\delta b}{a+b}=0.0000373,\quad \frac{\delta a+\delta b}{a+b}=-0.0000265;$$

$$\therefore\ \delta a=\frac{a+b}{2}\times 0.0000108=20890790\times 0.0000108=226 \text{ feet},$$

$$\delta b=-\frac{a+b}{2}\times 0.0000638=-1333 \text{ feet}.$$

Ex. 2. In the Indian arc, divided at Damargida, $\lambda=11^\circ 27' 33''=41253''$ from Kaliana to Damargida, and $\lambda'=9^\circ 53' 44''=35624''$ from Damargida to Punnæ: also $2m=47^\circ 34'$, $2m'=26^\circ 13'$.

$$\text{If } \delta\lambda=1'',\ \frac{\delta a-\delta b}{a+b}=-0.0000788,\quad \frac{\delta a+\delta b}{a+b}=-0.0001838,$$

$$\delta a=-\frac{a+b}{2}\times 0.0002626=-5486 \text{ feet},$$

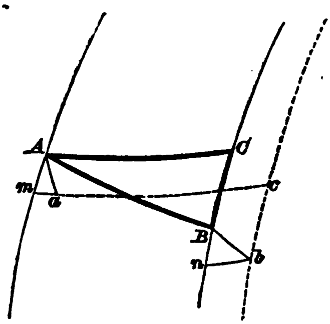
$$\delta b=-\frac{a+b}{2}\times 0.0001050=-219 \text{ feet}.$$

These are large quantities; and if they are so large for only 1″ of local attraction, they may be in fact much larger than this, without our having any means of knowing it. We have already shown (Art. 89) that there may be much larger deflections than 1″ without any visible cause to produce them. The calculations referred to in the last Article regarding the elliptical form of the equator are, therefore, not to be considered as trustworthy.

179. We are not aware that Arcs of Longitude or Azimuths have yet been made use of practically to find the Figure of the Earth. The Indian survey supplies ample materials for such a calculation: and we understand that it is in contemplation at some future time to take it up. The following determination of equations for finding the mean figure from any number of arcs of longitude, or of latitude, or of a mixed character, partly of longitude and partly of latitude, taking into account Local Attraction as before, may be useful for such an investigation. A method of using Azimuths is also given.

PROP. *To determine equations for calculating the semi-axes of the mean figure of the earth, by means of a mixed measured arc, partly of latitude and partly of longitude.*

180. The case of an arc partly of latitude and partly of longitude will be taken, because it includes those of an arc of latitude only and an arc of longitude only. Let A be the station in the triangulation to which all the other stations are referred, that is, from which the spherical coordinates to the other stations are measured. B any one of the other stations. AC, CB the spherical coordinates of longitude and latitude of B from A.



l, l' the observed latitudes of A and B,

λ, Λ .... amplitudes CB, AC in latitude and longitude.

Now the latitudes and longitudes observed at A and B are not the true latitudes and longitudes of those places, because they are affected by local attraction; but they are the true latitudes and longitudes of some other places a and b near A and B, such that Aa and Bb represent the differences of the observed from the true positions, and in fact measure the deflections of the apparent verticals at A and B from the true

verticals. Aa and Bb at the centre of the earth measure angles which are equal and opposite to the angular deflections of the plumb-line. Let those deflections be reckoned positive when they are to the north and west, and negative when to the south and east. Then

$$\frac{Am}{\text{mean rad. of earth}} = \text{deflection at } A \text{ in latitude } = t \text{ suppose}$$

$$\frac{Bn}{\text{mean rad. of earth}} = \ldots\ldots\ldots\ldots\ldots B \ldots\ldots\ldots\ldots = t' \quad ,,$$

$$\frac{ma}{\text{mean rad. } \cos l} = \ldots\ldots\ldots\ldots\ldots A \text{ in longitude} = T \quad ,,$$

$$\frac{nb}{\text{mean rad. } \cos l} = \ldots\ldots\ldots\ldots\ldots B \ldots\ldots\ldots\ldots = T' \quad ,,$$

Let S and s be the lengths which the Survey brings out by triangulation for the spherical coordinates AC, CB. These are independent of the effects of local attraction, as we shall see in Art. 203, 207. Suppose S laid on this spheroid, small angles x and z being added to the observed latitude and longitude of A, and x' and z' to those of B, in order to make the length, with the spherical coordinates thus altered, exactly fit the spheroid. By Art. 153

$$S = (\Lambda + z' - z) \cos (l + x) \{a + (a - b) \sin^2 (l + x)\};$$

$$\therefore\ z' - z = \frac{S}{a \cos l} - \Lambda \left(1 + \frac{a-b}{a} \sin^2 l\right) + \Lambda \tan l \,.\, x$$

$$= M' + A' \,.\, u + B' \,.\, v + \Lambda \tan l \,.\, x$$

where M', A', B' are numerical quantities, being functions of observed latitudes and longitudes and measured lengths; and the semi-axes are given by formulæ in Art. 159.

So also by Art. 159

$$x' - x = m' + \alpha' \,.\, u + \beta' \,.\, v,$$

where m', α', and β' are numerical quantities.

Quantities similar to z' and x' must be found for every station of the arc. Those values of u and v, and therefore of a and b, which make the sum of the squares of the corrections $z, z', z'' \dots x, x', x'' \dots$ a minimum, determine the particular spheroid which most nearly fits the arc and its several sub-divisions. These values of u and v we call U and V.

In this sum of squares all the quantities besides u and v are numerical quantities derived from observation and calculation, except x and z, the small additions which we give to the latitude and longitude of the reference station, by which we adjust the ends of the arc so as to make it fit the spheroid. If we do not tie down x and z by any condition, but treat them as quite arbitrary, we may no doubt get a spheroid somewhat nearer to the measured arc than we otherwise should. But by so doing we should be violating the conditions of the problem. For the problem to be solved is this. The arc and its various portions and stations are actual lines and points fixed on the earth. Conceive a set of countless spheroids drawn, each very nearly coinciding with the earth's surface, and having its centre in the earth's centre, and its axis in the same line with the earth's axis. That particular spheroid which most nearly coincides with the arc and its parts is called the mean spheroid. We are not at liberty arbitrarily to shift the arc from its real position with respect to the earth's centre and axis to some other position, so as to make the sum of the squares of the errors less than it otherwise would be; because though a spheroid may be thus got with which the arc would accord somewhat more closely, this would be attained by a sacrifice of truth. The line on the mean spheroid which would then represent the actual arc would not be its proper representative. The proper representative of the actual arc on the mean spheroid is its direct projection on its surface by normal lines drawn from the stations to the surface, or the nearest representative of that projection which we can get: and x and z must be so determined as to lead to this result. This has been already explained in Art. 162.

The mixed arc of latitude and longitude which we are considering may evidently be replaced by two arcs; one of

longitude from the first station to the meridian of the last, and the other an arc of latitude from that parallel to the last station along the meridian of that station. In treating of these we shall consider the points where the parallels of the various stations cut the meridian of the last station to be "the stations of the arc of latitude" marking its sub-divisions; and similarly of the arc of longitude. For the arc of latitude we have already shown (in Art. 162) that the proper value of x is

$$= n(v - V) + \frac{(t)}{i} - \frac{(m) + (\alpha)\,U + (\beta)\,V}{i}.$$

The same reasoning applies to the arc of longitude, with this simplification, that whereas it is a circle and not an ellipse, we must put $n = 0$; and we have

$$z = \frac{(T)}{i} - \frac{(M) + (A)\,U + (B)\,V}{i};$$

this is constant, whereas x involves v.

Proceeding as in Art. 163, differentiating the sum of the squares; viz.

$$\begin{aligned} &z^2 + (z + M' + A'u + B'v + \Lambda \tan l \,.\, x)^2 \\ &\quad + (z + M'' + A''u + B''v + \Lambda' \tan l \,.\, x)^2 \\ &\quad + \dots\dots\dots\dots \\ &+ x^2 + (x + m' + \alpha' u + \beta' v)^2 + \dots\dots \end{aligned}$$

with respect to x and v, we obtain these two equations:

$$\begin{aligned} 0 = {} & z(A) + (AM) + (A^2)\,U + (AB)\,V + (A\Lambda) \tan l \,.\, x \\ & + x(\alpha) + (\alpha m) + (\alpha^2)\,U + (\alpha\beta)\,V \\ 0 = {} & z(B) + (BM) + (BA)\,U + (B^2)\,V + (B\Lambda) \tan l \,.\, x \\ & + zn \tan l(\Lambda) + n \tan l \{(M\Lambda) + (A\Lambda)\,U + (B\Lambda)\,V\} \\ & \qquad + nx \tan^2 l \,.\, (\Lambda^2) \\ & + inx + n(m) + n(\alpha)\,U + n(\beta)\,V \\ & \qquad + x(\beta) + (\beta m) + (\beta\alpha)\,U + (\beta^2)\,V. \end{aligned}$$

In these substitute the values of x and z; they become

$$\left.\begin{aligned}
0 &= \left\{(A^2)+(\alpha^2)-\frac{(A)^2+(\alpha)^2+(A\Lambda)(\alpha)\tan l}{i}\right\}U \\
&\quad+\left\{(AB)+(\alpha\beta)-\frac{(A)(B)+(\alpha)(\beta)+(A\Lambda)(\beta)\tan l}{i}\right\}V \\
&\quad+(AM)+(\alpha m) \\
&\quad+\frac{(A)\{(T)-(M)\}+\{(\alpha)+(A\Lambda)\tan l\}\{(t)-(m)\}}{i}. \\
0 &= \left\{(BA)+(\beta\alpha)+n\tan l\,(A\Lambda)-\frac{(A)(B)+(\alpha)(\beta)}{i}\right. \\
&\qquad\left.-\tan l\,\frac{(B\Lambda)(\alpha)+n(\Lambda)(A)+n\tan l\,(\Lambda^2)(\alpha)}{i}\right\}U \\
&\quad+\left\{(B^2)+(\beta^2)+n\tan l\,(B\Lambda)-\frac{(B)^2+(\beta)^2}{i}\right. \\
&\qquad\left.-\tan l\,\frac{(B\Lambda)(\beta)+n(\Lambda)(B)+n\tan l\,(\Lambda^2)(\beta)}{i}\right\}V \\
&\quad+(BM)+(\beta m)+n\tan l\,(M\Lambda)+n(t) \\
&\qquad+\{(B)+n\tan l\,(\Lambda)\}\frac{(T)-(M)}{i} \\
&\qquad+\tan l\,\{(B\Lambda)+n\tan l\,(\Lambda^2)+(\beta)\}\frac{(t)-(m)}{i}.
\end{aligned}\right\}\ldots(1)$$

In these equations all the quantities are numerical, except U and V, which have to be found by their solution, and (T) and (t) the sums of the deflections at the several stations in longitude and latitude. This is the result for a mixed arc, that is, one partly of longitude and partly of latitude.

181. COR. 1. If all the capital letters, except U and V, are put equal to zero, that is, if the arc of longitude is annihilated, the equations for finding U and V become precisely those of Art. 163, viz.

$$\left.\begin{aligned}
0 &= \left\{(a^2) - \frac{(a)^2}{i}\right\} U + \left\{(a\beta) - \frac{(a)(\beta)}{i}\right\} V + (am) - \frac{(a)(m)}{i} \\
&\qquad + \frac{(a)}{i}(t). \\
0 &= \left\{(a\beta) - \frac{(a)(\beta)}{i}\right\} U + \left\{(\beta^2) - \frac{(\beta)^2}{i}\right\} V + (\beta m) - \frac{(\beta)(m)}{i} \\
&\qquad + \left(n + \frac{(\beta)}{i}\right)(t).
\end{aligned}\right\} \ldots (2)$$

182. Cor. 2. If all the small letters are made equal to zero the arc is reduced to one of longitude only, and the equations become

$$\left.\begin{aligned}
0 &= \left\{(A^2) - \frac{(A)^2}{i}\right\} U + \left\{(AB) - \frac{(A)(B)}{i}\right\} V \\
&\qquad + (AM) - \frac{(A)(M)}{i} + \frac{(A)}{i}(T) \\
0 &= \left[(AB) - \frac{(A)(B)}{i} + n \tan l \left\{(A\Lambda) - \frac{(A)(\Lambda)}{i}\right\}\right] U \\
&\qquad + \left[(B^2) - \frac{(B)^2}{i} + n \tan l \left\{(B\Lambda) - \frac{(B)(\Lambda)}{i}\right\}\right] V \\
&\qquad + (BM) - \frac{(B)(M)}{i} + n \tan l \left\{(M\Lambda) - \frac{(M)(\Lambda)}{i}\right\} \\
&\qquad + \{(B) + n \tan l (\Lambda)\} \frac{(T)}{i}.
\end{aligned}\right\} \ldots (3)$$

We may remark here, that if $n = 0$ in (2) and (3), that is, if the arcs of longitude and latitude are both circles and the spheroid becomes a sphere, these equations are of precisely the same form, which of course they ought to be.

Any number of arcs may be used in the calculation. If any one be an arc of latitude, then the first sides of equations (2) must be calculated for that arc; if any one be an arc of longitude, the first sides of equations (3) must be calculated; if any arc be of a mixed character equations (1) must be used. The results for the several arcs must be

added together and put equal to zero to form two final equations in U and V for the determination of the spheroid which most nearly fits all the arcs combined. Into these equations each arc of latitude will introduce an unknown quantity like (t); each arc of longitude an unknown like (T); each mixed arc two unknowns like (t) and (T).

PROP. *To prove that if the arcs are connected together by triangulation, then, however great their number, all the unknown quantities brought in by local attraction may be reduced to two only, like* (t) *and* (T).

183. S is the geodetic arc of longitude, Λ its observed or astronomical amplitude. Let Γ be the amplitude obtained from the geodetic arc when laid on the mean spheroid. Then by Art. 153

$$S = \Gamma \cos l \{a + (a - b) \sin^2 l\},$$

$$\therefore \Gamma = \frac{S}{a \cos l} - \Gamma \frac{a-b}{a} \sin^2 l = \frac{S}{a \cos l} - \Lambda \frac{a-b}{a} \sin^2 l,$$

the last term being small Λ may be put for Γ. Introducing M', A', B' from Art. 180, and putting U and V for u and v because we are using the *mean* spheroid, we obtain

$$M' + A'U + B'V = \Gamma - \Lambda,$$

that is, the excess of the geodetic amplitude over the astronomical amplitude. This, it will be seen in Art. 207, equals the difference of the local deflections in longitude at the extremities of the arc, $= T' - T$, the deflection of the plumb-line to the west being reckoned positive.

Similarly $\quad M'' + A''U + B''V = T'' - T$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$M^{(r)} + A^{(r)}U + B^{(r)}V = T^{(r)} - T$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\therefore (M) + (A)U + (B)V = T' + T'' \dots\dots T^{(i)} - (i-1)T$$
$$= (T) - iT,$$

also $$iM^{(r)} + iA^{(r)}U + iB^{(r)}V = iT^{(r)} - iT;$$

$$\therefore (T) = i(T^{(r)} - M^{(r)} - A^{(r)}U - B^{(r)}V) + (M) + (A)U + (B)V,$$

in this all the quantities besides U and V are numerical except (T) and $T^{(r)}$; and by this formula the unknown (T) can be replaced by $T^{(r)}$ the unknown local deflection at any particular station on the arc. Again

$$T^{(s)} = T^{(r)} + M^{(s)} - M^{(r)} + (A^{(s)} - A^{(r)})\,U + (B^{(s)} - B^{(r)})\,V,$$

by which formula the unknown $T^{(r)}$ may be replaced in the equations in U and V by $T^{(s)}$ the unknown local deflection at any other station of the arc.

The same things may be proved regarding (t): for if γ be the geodetic amplitude of latitude when the arc is laid on the mean spheroid; then by Art. 150

$$2s = (a+b)\gamma - 3(a-b)\sin\gamma\cos(l+l');$$

$$\therefore \gamma = \frac{2s}{a+b} + 3\frac{a-b}{a+b}\sin\lambda\cos(l+l'),$$

λ being put for γ in the last term as it is small. Introducing m', α', β' from Art. 180, we have

$$m' + \alpha' U + \beta' V = \gamma - \lambda,$$

that is, the excess of the geodetic over the astronomical amplitude. But this (see Arts. 203, 204) equals the difference of the local deflections at the extremities of the arc $= t' - t$; and the same process may now be followed as for T, T' in the last article.

184. The use which can be made of these results is the following:—If a gridiron of arcs be used, however complicated, the unknowns (T) and (t) of any arc composing it may be replaced by the unknowns $T^{(r)}$ and $t^{(r)}$ at the station where it joins or crosses another arc. These unknowns may be replaced by unknowns $T^{(s)}$ and $t^{(s)}$ at another station along this second arc, where a third arc branches off. In this way all the unknowns in the final equations in U and V may be reduced to two, $T^{(c)}$ and $t^{(c)}$, at some common station adopted as a reference station.

185. If, then, the arcs used are all connected together by triangulation, the final equations in U and V will involve one or two unknown quantities, according as arcs of longitude or latitude only are used for the calculation, or only mixed arcs. If a number of arcs are used for the calculation, some connected by triangulation and some not, the final equations in U and V will contain one unknown for each independent arc of latitude or longitude, and two for each independent mixed arc. All these quantities being unknown, the axes of the mean spheroid are also unknown. This arises from taking account of local attraction, as it is on this that the unknowns depend.

186. As in Art. 168 so now the only method which we can suggest for getting over this difficulty, besides *neglecting* local attraction, is the following:—

Make independent calculations for the three long arcs of meridian, the Anglo-Gallic, the Russian, and the Indian Great Arc, by means of equations (2), and suppose that (t_1), (t_2), (t_3) are the three unknowns; we thus obtain three sets of semi-axes

$$a_1 = P_1 + Q_1(t_1), \qquad a_2 = P_2 + Q_2(t_2), \qquad a_3 = P_3 + Q_3(t_3),$$
$$b_1 = R_1 + S_1(t_1), \qquad b_2 = R_2 + S_2(t_2), \qquad b_3 = R_3 + S_3(t_3).$$

Then make the calculation for the Indian Gridiron (including the Great Arc of Meridian or not) by means of equations (1). This will involve the former unknown (t_3) as the Great Arc of Meridian is connected by triangulation with the Gridiron, and another unknown (T_4), and we shall have

$$a_4 = P_4 + Q_4(t_3) + Q'_4(T_4),$$
$$b_4 = R_4 + S_4(t_3) + S'_4(T_4),$$

P, Q, R, S, Q', S' are all numerical quantities. By putting $a_1 = a_2 = a_3 = a_4$, $b_1 = b_2 = b_3 = b_4$ we obtain six equations connecting four unknowns; the most probable values of which may be found by the method of least squares; and these values being substituted in the expressions for $a_1 b_1$, $a_2 b_2$, ... we at once see how nearly the figures of the earth thus obtained agree with each other, and the mean of them may be regarded as the Mean Figure.

We might omit the Anglo-Gallic or Russian arc: we should then have four equations and three unknowns; and the same process followed. But the result would give a figure partaking more of the peculiarities of the Indian part of the globe.

If it is desired to find the spheroid which most nearly represents the surface of India, the Indian Great Arc of latitude, the Great mixed Arc from Karachi to Calcutta, and the Gridiron (including or not one or both of the former) might be calculated separately. These would lead to four equations involving two unknowns (t) and (T), and the nearest values may be found by the method of least squares as before.

PROP. *To give a method for introducing Azimuths into the calculation.*

187. It has been thought, that as azimuths are frequently observed, as well as calculated, during survey operations, they might be brought to bear upon the problem of the figure of the earth. The following calculation, which shows how this can be done, may help to determine whether they are likely to add much or not to the trustworthiness of the final results. It will be seen that, as they are merely angles and do not depend on lengths, they can assist in determining only the ratio of the axes, and not the axes themselves, and therefore only the ellipticity.

Let $$x^2 + y^2 + (1 + 2\epsilon')\,z^2 = a'^2 \dots\dots\dots\dots(1)$$

be the equation to the ellipse which generates the spheroid on which the survey calculations are made. XYZ being general coordinates,

$$\left.\begin{aligned} X - x &= (1 - 2\epsilon')\frac{x}{z}(Z - z) \\ Y - y &= (1 - 2\epsilon')\frac{y}{z}(Z - z) \end{aligned}\right\}$$ are equations to the normal at point $A\,(xyz)$: (see diagram Art. 180).

$0 = yX - xY$, equation to meridian plane through A...(2),

$Z - z = A\,(X - x) + B\,(Y - y)$, equation to normal plane at A through B ....(3),

A and B being therefore determined by the following conditions:—

$z'-z=A(x'-x)+B(y'-y)$, $x'y'z'$ being co-ordinates to B,

and $(1+2\epsilon')z=Ax+By$, $\because$ the plane includes the normal...(4),

$$\therefore z'+2\epsilon'z=Ax'+By', \text{ by addition;}$$

$$\therefore A=\frac{zy'-z'y+2\epsilon'z(y'-y)}{xy'-x'y}, \quad B=\frac{xz'-x'z-2\epsilon'z(x'-x)}{xy'-x'y},$$

cos azimuth = cos angle between planes (2) and (3)

$$=\frac{Ay-Bx}{\sqrt{1+A^2+B^2}\sqrt{x^2+y^2}};$$

$$\therefore \cot\text{ azimuth}=\frac{Ay-Bx}{\sqrt{x^2+y^2+(Ax+By)^2}}=\frac{Ay-Bx}{\sqrt{a'^2+2\epsilon'z^2}} \text{ by (1) (4)}$$

$$=\frac{z(1+2\epsilon')(xx'+yy'+zz')-a'^2z'-2\epsilon'z(a'^2-z^2)}{a'(xy'-x'y)}\left(1-\epsilon'\frac{z^2}{a'^2}\right).$$

Let r and θ be the polar co-ordinates to the elliptic meridian through A;

$$\therefore x=r\cos\theta\cos L=a'\cos l\cos L(1-\epsilon'\sin^2 l),$$
$$y=r\cos\theta\sin L=a'\cos l\sin L(1-\epsilon'\sin^2 l),$$
$$z=r\sin\theta \qquad =a'\sin l(1-2\epsilon'-\epsilon'\sin^2 l).$$

Similarly for $x'y'z'$.

Let ζ be the azimuth of B at A for ellipticity ϵ',

ζ_0 .... a sphere,

ζ' and ζ'_0 the same angles of A at B,

Σ the length of the arc AB referred to a sphere rad. = 1;

$$\therefore \cos\Sigma=\sin l\sin l'+\cos l\cos l'\cos\Lambda,$$

$ll'LL'$ are all along the *observed* latitudes and longitudes. We will in the first instance neglect local attraction.

Now $\frac{xx' + yy' + zz'}{a'^2} = (\sin l \sin l' + \cos l \cos l' \cos \Lambda)$

$$\times \{1 - \epsilon' (\sin^2 l + \sin^2 l')\} - 4\epsilon' \sin l \sin l',$$

$$\frac{xy' - x'y}{a'^2} = \cos l \cos l' \sin \Lambda \, \{1 - \epsilon' (\sin^2 l + \sin^2 l')\};$$

$$\therefore \cot \zeta = \frac{\sin l (\sin l \sin l' + \cos l \cos l' \cos \Lambda) - \sin l'}{\cos l \cos l' \sin \Lambda}$$

$$- 2\epsilon' \frac{\sin^3 l \cos \Sigma - \sin l' (1 - 2 \sin^2 l) + \sin l \cos^2 l}{\cos l \cos l' \sin \Lambda}$$

$$= \cot \Lambda \sin l - \operatorname{cosec} \Lambda \cos l \tan l'$$

$$- \frac{2\epsilon'}{\cos l \cos l' \sin \Lambda} \{\sin^3 l \cos \Sigma + \sin l \cos^2 l - \sin l' (1 - 2 \sin^2 l)\}.$$

Hence for a sphere

$$\cot \zeta_0 = \cot \Lambda \sin l - \operatorname{cosec} \Lambda \cos l \tan l' \dots\dots\dots\dots\dots (5),$$

$\therefore \cot \zeta = \cot \zeta_0 - \epsilon'. C$, putting the coefficient of $\epsilon' = C$;

$$\therefore \zeta - \zeta_0 = \tan^{-1} \tan (\zeta - \zeta_0) = \tan^{-1} \frac{\epsilon' C}{1 + \cot^2 \zeta_0} = \frac{\epsilon' C}{1 + \cot^2 \zeta_0}.$$

But $1 + \cot^2 \zeta_0 = 1 + \cot^2 \Lambda \sin^2 l + \operatorname{cosec}^2 \Lambda \cos^2 l \tan^2 l'$

$$- 2 \cot \Lambda \operatorname{cosec} \Lambda \sin l \cos l \tan l'$$

$$= \frac{1 - (\cos l \cos l' \cos \Lambda + \sin l \sin l')^2}{\sin^2 \Lambda \cos^2 l'} = \frac{\sin^2 \Sigma}{\sin^2 \Lambda \cos^2 l'} \dots\dots (6).$$

Hence we have finally

$$\zeta = \zeta_0 + \epsilon' P,$$

where $P = \frac{2 \sin \Lambda \cos l'}{\sin^2 \Sigma \cos l}$

$$\times \{\sin^3 l \cos \Sigma + \sin l \cos^2 l - \sin l' (1 - 2 \sin^2 l)\} \dots\dots (7).$$

188. We have hitherto supposed that there is no local attraction. We must now allow for this. Instead of l, l', Λ, we must substitute $l + t$, $l' + t'$, $\Lambda + T' - T$, and then

$$\zeta_0 \text{ becomes } \zeta_0 + \frac{d\zeta_0}{dl} t + \frac{d\zeta_0}{dl'} t' + \frac{d\zeta_0}{d\Lambda} (T' - T) \dots\dots\dots (8)$$

As P is multiplied by the ellipticity, and we neglect small quantities of the second order, we need not alter the angles in P.

Now by (5) and (6)

$$\frac{d\zeta_0}{dl} = -\sin^2\zeta_0 (\cot\Lambda\cos l + \operatorname{cosec}\Lambda\sin l\tan l')$$

$$= -\frac{\sin\Lambda\cos l'\cos\Sigma}{\sin^2\Sigma} = E \text{ suppose};$$

$$\frac{d\zeta_0}{dl'} = \sin^2\zeta_0 \operatorname{cosec}\Lambda\cos l\sec^2 l' = \frac{\sin\Lambda\cos l}{\sin^2\Sigma} = F,$$

$$\frac{d\zeta_0}{d\Lambda} = \sin^2\zeta_0\operatorname{cosec}^2\Lambda\,(\sin l - \cos\Lambda\cos l\tan l')$$

$$= \frac{\cos l'}{\sin^2\Sigma}\{\sin l\cos l' - \tan l'(\cos\Sigma - \sin l\sin l')\}$$

$$= \frac{\sin l - \sin l'\cos\Sigma}{\sin^2\Sigma} = G.$$

Hence ζ_0, corrected for local deflection (8), becomes

$$\zeta_0 + Et + Ft' + G(T' - T)$$

and the true azimuth ζ

$$= \zeta_0 + Et + Ft' + G(T' - T) + \epsilon P \quad \text{.................(9)}$$

in which the *observed* values of l, l', Λ, are used.

If accentuated letters are used for B, as unaccentuated are for A, then

$$\zeta' = \zeta_0' + E't + F't' + G'(T' - T) + \epsilon P'$$

where $E' = -\frac{\cos l}{\cos l'}E$, $F' = -\frac{\cos l'}{\cos l}F$, $G' = \frac{\sin l' - \sin l\cos\Sigma}{\sin l - \sin l'\cos\Sigma}G$

$$P' = -\frac{2\sin\Lambda\cos l}{\sin^2\Sigma\cos l'}\{\sin^3 l'\cos\Sigma + \sin l'\cos^2 l' - \sin l(1 - 2\sin^2 l)\}$$

l and l' being interchanged in the formulæ, and therefore $-\Lambda$ put for Λ.

188*. Suppose ϵ'' is the ellipticity of the spheroid which the particular azimuth at A fits exactly when applied to it. Then by (9)

$$\epsilon'' = \frac{\{\zeta - Et - Ft' - G(T' - T)\} - \zeta_0}{P}.$$

The terms enclosed in brackets in the numerator are the true azimuth altered by local deflection at the two extremities of the arc AB. It might, therefore, at first sight be supposed, that we may substitute for these terms the *observed* azimuth, and, as the other terms in the expression for ϵ'' can be calculated, that we can find the ellipticity ϵ'' without knowing the local attractions: and, obtaining like values of ϵ'' from all other observed azimuths and taking their mean, thus obtain the ellipticity of the mean figure of the earth. This, however, is not the case. The station B, as seen from A, is not affected by the local deflection at B; and the station A, as seen from B, is not affected by the local deflection at A. Hence the *observed* azimuth at A is

$$\text{not } \zeta - Et - Ft' - G(T' - T) \text{ which} = \zeta_0 + \epsilon''P,$$
$$\text{but } \zeta - Et + Gt \text{ which} = \zeta_0 + Ft' + GT' + \epsilon''P.$$

If A and B are not mutually visible from each other, yet as they are connected by a chain of horizontal triangles, and the stations at their angular points are visible one from another in succession, the calculated bearing of B at A, and of A at B, will be as much independent of the local deflection at B, and at A, as if they were visible from each other.

189. In the last Prop., the arcs AC, BC (see diagram, Art. 180), were supposed to be laid upon the variable spheroid of which the semi-axes are a and b, after the latitudes and longitudes of A and B are increased by small quantities xx' and zz', so as to make the arcs fit the spheroid: and we have shown how $x'x''\ldots$, $z'z''\ldots$ for the successive arcs into which the whole arc is divided, can be found in terms of x and z; and also how x and z are determined in terms involving the local deflections and known quantities. We must now devise means for introducing the observed azimuths, as well as the measured arcs, into the sum of the

squares which is to be made a minimum for calculating the mean figure. To do this we must find the difference between the azimuth as observed and the corresponding azimuth on the variable spheroid, and add the square of this, and also the squares of all similar quantities for the other azimuths, to the sum of the squares, which is to be made a minimum; and we shall get a spheroid somewhat closer to the whole data than before.

Let $\epsilon = (a-b) \div a$ be the ellipticity of the variable spheroid, the axes of which are given in Art. 159;

$$\therefore \epsilon = \frac{v}{15000},$$

and by (9) $\quad \zeta = \zeta_0 + Et + Ft' + G(T' - T) + \epsilon P.$

Suppose Z and Z' are the *observed* azimuths at A and B.

Then $\quad Z = \zeta_0 + Ft' + GT' + \epsilon' P \ldots\ldots\ldots\ldots\ldots(10),$

that is, ζ_0, being calculated from the *observed* latitudes and longitudes of A and B and therefore involving local deflections for both A and B, must be corrected for local deflection at B, as local attraction at B has no effect on the position of B as seen from A. The ellipticity used is ϵ', that of the Survey, because the connection of B with A is calculated with that ellipticity in the Survey operations.

So also $\quad Z' = \zeta_0' + E't - G'T + \epsilon' P' \ldots\ldots\ldots\ldots (11).$

Hence $\quad \zeta - Z = Et - GT + (\epsilon - \epsilon')P$

and $\zeta' - Z' = F't' + G'T' + (\epsilon - \epsilon')P'.$

These are the differences between the observed azimuths at A and B and those on the variable spheroid. We may eliminate T and T' by (10), (11); and these variations become, substituting for ϵ,

$$(Z' - \zeta_0')\frac{G}{G'} - \frac{GP' + G'P}{G'}\epsilon' + \frac{EG' - E'G}{G'}t + \frac{P}{15000}v,$$

$$(Z - \zeta_0)\frac{G'}{G} - \frac{G'P + GP'}{G}\epsilon' + \frac{F'G - FG'}{G}t' + \frac{P'}{15000}v.$$

Call these, and those for the other arcs,

$$\begin{array}{ll} \mu_1 + p_1 t + q_1 v & \nu_1 + r_1 t' + s_1 v \\ \mu_2 + p_2 t + q_2 v & \nu_2 + r_2 t'' + s_2 v \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \end{array}$$

All the quantities are constant except v. The sum of the squares differentiated, and V put for v, gives

$$\begin{aligned} &q_1(\mu_1 + p_1 t + q_1 V) + s_1(\nu_1 + r_1 t' + s_1 V) \\ &+ q_2(\mu_2 + p_2 t + q_2 V) + s_2(\nu_2 + r_2 t'' + s_2 V) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

$$\text{or } (q\mu) + (qp)t + (q^2)V + (s\nu) + (srt) + (s^2)V,$$

$$\text{or } (q\mu) + (s\nu) + (qp)t + (srt) + \{(q^2) + (s^2)\}V.$$

This must be added to the second side of the second of the equations obtained by differentiation at the beginning of Art. 180, which is the same as adding to the second side of the second of equations (1). It will increase the coefficient of V, and add some terms to the part independent of U and V. It introduces two unknown quantities, viz. t and (srt). The first can be obtained in terms of (t), already in equations (1), by the process explained in Arts. 183—185. The second (srt) is the sum of all the local deflections in latitude, after that at A, each multiplied by the numerical quantity represented by sr before the addition takes place: this also can be obtained in terms of (t) by the same process as before. Hence the introduction of azimuths into the problem brings in no more unknown quantities than before. Where they are independent and belong to other arcs not before in the problem, one new unknown quantity is brought in by each set of azimuths.

190. Suppose, for example, it is desired to find the mean figure of the Continent of India; and we connect the azimuths with the Gridiron, and take also the Great Arc of meridian through Cape Comorin, and the Great mixed Arc through Karachi. We shall have four equations and two unknown quantities (t), (T): and the best values of these, and of the axes, can be found by least squares.

Besides thus finding the mean Figure of the Indian Continent, the mean Figure of the Earth may be also found by calculating the figure separately for the Anglo-Gallic, and the Russian arcs of meridian, the mixed Indian arc from Karachi to Calcutta, and the Indian Gridiron including the Great Arc through Cape Comorin. This will give six equations and four unknowns to be determined by least squares.

The mean form of the Indian Continent would then be compared with the mean figure of the earth.

191. These formulæ have not yet been reduced to figures, inasmuch as the amounts of local attraction are not known. We must satisfy ourselves, therefore, with the results of Arts. 168—173, according to which

$$a = 20926184, \quad b = 20855304, \quad \text{and } \epsilon = \frac{1}{295\cdot 2},$$

$$(t_1) = 0.610'', \quad (t_2) = -0'.031, \quad \text{and } (t_3) = 0''.007,$$

$$\text{also } U = -0.3562, \quad V = 0.8934.$$

When these values are substituted in the formulæ for the corrections of latitude alluded to in Art. 164, Cor. 1, we have the local deflections at all the stations on the three arcs.

For the Anglo-Gallic, Russian, and Indian arcs (Art. 169) $i = 34$, 13, and 8. Hence

$$\frac{(t_1)}{i} = 0''.018, \quad \frac{(t_2)}{i} = -0''.003, \quad \frac{(t_3)}{i} = 0.''001.$$

The mean ellipse is obtained in the Ordnance Survey Volume by a comparison of these three great arcs together with five very short arcs: see Art. 175. These five arcs are so very short that they cannot sensibly affect the value there deduced for x for each of the three great arcs above-named. If then (since the mean ellipse of the Ordnance Survey is the same as ours) we add the values of $(t) \div i$ above deduced to the values of x in the Ordnance Survey Volume (see Art. 164), we shall have the values of x for the three great arcs we are using, when local attraction is taken account of. The results are given in the following table for all the stations of the arcs, counting always from the south end of each arc: the angles

are the corrections which must be added to the observed latitudes of the stations to make them fit the mean ellipse, and therefore are the local deflections of the plumb line: see Art. 164, Cor. 2.

Anglo-Gallic Arc.		20	− 0″.901	4	− 2″.029
1	+ 2″.602	21	− 2 .161	5	+ 0 .272
2	+ 4 .826	22	+ 1 .270	6	− 1 .814
3	+ 1 .470	23	+ 0 .437	7	+ 2 .530
4	− 0 .407	24	− 1 .167	8	− 1 .440
5	− 3 .380	25	− 1 .873	9	− 0 .619
6	− 0 .783	26	− 0 .840	10	− 0 .971
7	− 1 .210	27	+ 1 .868	11	+ 3 .809
8	− 2 .885	28	− 0 .497	12	− 1 .407
9	− 1 .346	29	+ 0 .403	13	− 0 .019
10	− 1 .867	30	− 1 .338		
11	− 1 .589	31	+ 0 .252	*Indian Arc.*	
12	− 1 .810	32	+ 2 .344	1	− 1″.333
13	− 1 .269	33	+ 2 .203	2	− 1 .817
14	+ 2 .904	34	+ 0 .220	3	+ 3 .722
15	+ 1 .660			4	− 1 .948
16	+ 1 .171	*Russian Arc.*		5	+ 0 .051
17	+ 0 .709	1	− 2″.429	6	+ 2 .678
18	+ 0 .049	2	+ 1 .258	7	− 3 .155
19	+ 1 .550	3	+ 2 .822	8	+ 1 .811

They all come out remarkably small, none of them at all to be compared with the large deflections caused by the Himalayas and the Ocean in India. Thus even at the two extreme stations of the Great Arc of India beginning with Cape Comorin they are only − 1″.333 and + 1″.811. And it is curious that out of 13 coast-stations, in 7 of them what deflection there is is *towards* the sea, viz. in the Anglo-Gallic Arc, Nos. 10, High Port Cliff; 11, Week Down; 12, Boniface Down; 13, Dunnose; 23, Burleigh Moor; 34, Saxaford: and in the Russian Arc, No. 13, Fuglenæs.

PROP. *To deduce from the previous calculation some probable conclusions regarding the Constitution of the Earth's Crust.*

192. An hypothesis which the foregoing calculations seem to point to is this, That, supposing that the earth was once in a fluid state, as it became solid it contracted unequally, leaving mountains where it contracted least and ocean-beds where it contracted most. This we shall now endeavour to show from the data.

The first thing to be observed in the results given in the last paragraph is the very small amount of the resultant deflections at the stations of the Indian Arc; whereas the effect of the Ocean and the Mountains has been shown to be very large. This shows that the effect of variations of density in the crust must be very great, in order to bring about this near compensation. In fact the density of the crust beneath the mountains must be less than that below the plains, and still less than that below the ocean-bed. If solidification from a fluid state commenced at the surface, the amount of contraction in the solid parts beneath the surface of the mountain-region has been less than in the parts beneath the sea-bed. In fact, it is this unequal contraction which appears to have caused the hollows in the external surface which have become the basins into which the waters have flowed to form the ocean. As the waters flowed into the hollows thus created, the pressure on the ocean-bed would be increased, and the crust, so long as it was sufficiently thin to be influenced by hydrostatic principles of floatation, would so adjust itself that the pressure on any *couche de niveau* of the fluid should remain the same. At the time that the crust first became sufficiently thick to resist fracture under the strain produced by a change in its density, that is, when it first ceased to depend for the elevation or depression of its several parts upon the principles of floatation, the total amount of matter in any vertical prism, drawn down into the fluid below to a given distance from the earth's centre, had been the same through all the previous changes. After this, any further contraction or any expansion in the solid crust would not alter the amount of matter in the vertical prism, except where there was an ocean; in the case of greater contraction under an ocean than elsewhere, the ocean would become deeper and the amount of matter greater, and in case of a less contraction or of an expansion of the crust under an

ocean, the ocean would become shallower and water would flow away, or the amount of matter in the vertical prism would become less than before. It is not likely that expansion and contraction in the solid crust would affect the arrangement of matter in any other way. That changes of level do take place, by the relative rising and sinking of parts of the surface, is a well-established fact, which rather favours these theoretical considerations. But they receive, we think, great support from the other fact, that the large effect of the ocean and of the mountains almost entirely disappears from the resultant deflections brought out by the calculations for the stations of the Indian Arc.

That part of the theory which shows that the wide ocean has been collected on parts of the earth's surface where hollows have been made by the contraction and therefore increased density of the crust below, is well illustrated by the existence of a whole hemisphere of water, of which New Zealand is the pole, in stable equilibrium. Were the crust beneath only of the same density as that beneath the surrounding continents, the water would be drawn off by attraction and not allowed to stand in the undisturbed position it now occupies.

193. We have, in what goes before, supposed that, in solidifying, the crust contracts and grows denser, as this appears to be most natural, though, after the solid mass is formed, it may either expand or contract, according as an accession or diminution of heat may take place. If, however, in the process of solidifying, the mass becomes lighter, the same conclusion will follow—the mountains being formed by a greater degree of expansion of the crust beneath them, and not by a less contraction, than in the other parts of the crust. It may seem at first difficult to conceive how a crust could be formed at all, if in the act of solidification it becomes heavier than the fluid on which it rests; for the equilibrium of the heavy crust floating on a lighter fluid would be unstable, and the crust would sooner or later be broken through, and would sink down into the fluid, which would overflow it. If, however, this process went on perpetually, the descending crust, which was originally formed by a loss of heat radiated from

the surface into space, would reduce the heat of the fluid into which it sank, and after a time a thicker crust would be formed than before, and the difficulty of its being broken through would become greater every time a new one was formed. Perhaps the tremendous dislocation of stratified rocks in huge masses with which a traveller in the mountains, especially in the interior of the Himalaya region, is familiar, may have been brought about in this way. The catastrophes, too, which geology seems to teach have at certain epochs destroyed whole species of living creatures, may have been thus caused, at the same time breaking up the strata in which those species had for ages before been deposited as the strata were formed. These phenomena must now long have ceased to occur, at any rate on a very extensive scale, as Mr Hopkins' and Sir William Thomson's investigations prove that the crust is very thick. See Arts. 133, 136. There must undoubtedly still be enclosed seas, or pockets (as geologists call them), of fluid matter in the crust itself, in order to account for volcanic eruptions. Volcanoes are in fact merely ulcers in the upper part of the crust, often connected by subterranean seas or channels of lava. But within the memory and traditions of man no such tremendous fractures of the crust have occurred as we see in the Himalayas and other mountain ranges. The thickness of the crust has for ages been too great for this. When the crust was thin the horizontal force of compression must have been enormous, as the fluid mass within went on contracting by loss of heat, enough to bend and distort any rocks *.

194. The existence of mountain regions used to be attributed to the action of upheaving forces from below. But we are disposed to attribute their appearance to the steady contraction, which must have been going on ever since solidification began. The horizontal force of compression thrown

* The Rev. O. Fisher shows that the horizontal force of compression thrown into a stratum at the earth's surface, by the shrinking of the parts below it, will equal the weight of a piece of rock of the same section as the stratum and 2000 miles long, enough to crumple up and distort any rocks. *Cambridge Philosophical Transactions*, Vol. XI. Part III.

into the solid parts by any contraction of the nucleus would cause the crust to yield in the weaker parts, and would have been more than sufficient to produce the most stupendous fractures, flexures, intrusions, upheavals of the surface in anticlinal lines, and all the phenomena which mountain regions present. As contraction went on in subsequent ages, no doubt the partial sinking of these huge masses might, certainly while the crust was thin, cause corresponding though subordinate upheavals in neighbouring parts of the crust. But the main part of the process which has been going on from the beginning has, no doubt, been contraction and subsidence; parts have been crumpled up into peaks, ridges, anticlinal lines; but other and more extensive parts have sunk to a lower level than before. Even the relative rise which has been noticed in some continents may really be the result of an actual sinking of the other parts of the earth's surface arising from constant contraction of the mass. It is considered that Scandinavia is rising at the rate of 2½ feet in a century. It may be that this is the average rate at which the earth's surface is shrinking towards the earth's centre, leaving Scandinavia relatively behind, as if it really rose. And so of other parts which appear to be rising. This rate of shrinking (two feet and a half in a century all over the earth) would diminish the length of the day during 6000 years by only 3-5ths of a second of time. And therefore even a larger shrinking, such as to leave 2½ feet in a century as the *difference* only of the shrinking of the several parts of the surface, would not be likely to produce a difference in the length of the day within 6000 years which we could by any means detect. There have been various ways suggested in which the force of compression and distortion could have been brought into play, besides that already mentioned, the contraction of the fluid nucleus and crust. They all suppose that the crust was at the time thin; and therefore they conspire to show the great antiquity of the earth; these grand phenomena which a traveller in the mountains witnesses must have occurred ages upon ages ago. Sir John Herschel many years since pointed out that the gradual transfer of solid matter by the constant action of rivers to the bed of the sea for the formation of future continents must, when the crust was thin, have

tended to derange the equithermal surfaces below, and both from this cause and the additional weight brought to bear upon that part of the crust must at times have produced a sudden sinking and breaking up. Mr Hall, of New York, thinks that he traces this action in the circumstances of the Alleghany Mountains*. He says that the aggregate thickness of the formations in the mountain region is 40,000 feet; whereas in the neighbourhood where no disturbance has taken place, the thickness is only 4000 feet. Here, then, the greatest disturbance has taken place, and no doubt frequently, where the largest deposits and sinking occurred. In the act of sinking the (arched) crust of any region would have some way to sink before it reached the chord; and compression, fracture, upheaval in anticlinal lines, distortion, penetration, sliding, and doubling, would be taking place all that time. Professor Rogers advances this theory; that the strata of the region affected were at times under great tension, from the expansion of solid matter and vapours below: that this found sudden vent and relief through a fissure; and down plunged the part of the crust most affected, and threw the fluid, and therefore the crust, into violent vibration, which caused fissures and other phenomena. This, no doubt, is the process in the lesser phenomena of volcanic eruption and earthquakes. Mr Hopkins, in some excellent papers on the Weald, called the same kind of force into use, and showed how the region affected would, on mechanical principles, split up into two systems of parallel fissures at right angles to each other, a theory quite borne out by the facts of that region. These various modes of action may no doubt have occurred. But contraction must have been the most powerful agent. All these modes of explaining the fracture of the earth's crust appertain to a time when the crust was comparatively thin, which it now is not; and in consequence of this we infer that these phenomena, which give the earth's surface its grand and beautiful variety, must have been produced many ages ago: and even before that subsequent process of further soli-

* See an interesting Appendix to Vol. III. of the *Memoirs of the Geological Survey of India*, by Mr Medlicott. Also a paper in the *Geological Society Journal*, February 1868, by the same author, in which the disturbance of the newer beds at the foot of the Alps and Himalayas is attributed to the sinking of the vast mountain mass.

dification and unequal contraction took place which, according to the hypothesis now advocated, have given the earth its present outward form. Contraction is no doubt still going on, but at a rate only perceptible in such phenomena as those of Scandinavia. Vast breakings up of the crust have long ceased.

195. The circumstance already noticed, that at seven coast-stations out of thirteen the deflection is towards the sea, seems to bear testimony to the truth of the hypothesis here advocated, that the crust below the ocean must have undergone greater contraction than other parts. The deflection towards the land at the other six coast-stations can of course easily be understood without at all calling in question the theory. The proximity of the land may easily be conceived sufficient to counteract any effect of the more distant parts of the crust below the ocean. It is the fact of even *some* of the deflections being towards the sea, that bears testimony to the theory, while the others offer no argument to the contrary. These coast-stations, therefore, rather confirm the theory so remarkably suggested by the facts brought to light in India, viz. that mountain-regions and oceans on a large scale have been produced by the contraction of the materials, as the surface of the earth has passed from a fluid state to a condition of solidity—the amount of contraction beneath the mountain-region having been less than that beneath the ordinary surface, and still less than that beneath the ocean-bed, by which process the hollows have been produced into which the ocean has flowed. In fact, the testimony of these coast-stations may be pushed further in favour of the theory, as they seem to indicate, by *excess* of attraction towards the sea, that the contraction of the crust beneath the ocean has gone on increasing in some instances still further since the crust became too thick to be influenced by the principles of floatation, and that an additional flow of water into the increasing hollow has increased the amount of attraction upon stations on its shores*.

* The first part of this theory apparently confirms Mr Airy's hypothesis (*Phil. Trans.* 1855, p. 101). But his reasoning is based on the crust being thin—so thin as to be influenced in its position by the fluid below; which cannot be admitted. Also see a note to a paper by the author in the *Philosophical Transactions* for 1871.

196. By the method given in Arts. 79, 80, the author has applied the results of Pendulum Observations, recently made in India, to test this hypothesis, in a paper communicated to the Royal Society in 1871. The result is as recorded in the accompanying table. The numbers given are the last figures in seven places of decimals representing the ratio of these differences to gravity itself. The decimal point and ciphers are left out for convenience.

Stations.	Differences of gravity with reference to that at Punnæ.			
	Relative effects of local attraction deduced from Pendulum Observations.	Residual errors after correction by		
		Dr Young.	This Hypothesis m = 50	m = 100
Indian Arc.				
Punnæ	...	...	...	...
Bangalore	+384	−562	− 78	−557
Damargida	−323	−926	−455	−584
Kalianpur	+341	−208	+338	+315
Kaliana	−707	−957	+ 69	+320
Coast.				
Punnæ	...	...	...	...
Alleppy	+302	+314	+331	+360
Mangalore	−166	−154	−122	− 79
Madras	−197	−192	−138	− 78
Cocanada	+142	+153	+216	+291
Ocean.				
Minicoy Is.	+894	+906	+ 31	+102

The first column, which is derived from the pendulum observations, the effects of height and latitude having been eliminated in order to make the comparison, shows that at the five nearly equally distant stations we have chosen from the Indian Arc of Meridian, the effects of local attraction relatively to the south extremity of the arc are alternately in excess and defect, that at the next station (Bangalore) being

in excess. The greatest effects of all the places considered are at Kaliana, in defect, and in Minicoy Island (250 miles west of Punnæ) in excess, although it is in the open ocean.

It will be observed that the usual method of correcting for local attraction (that is, Dr Young's) aggravates the errors on the Arc of Meridian and in Minicoy, instead of reducing them. Neither method, Dr Young's or this hypothesis, affects the coast-stations much. Those parts need better surveying. But in the Arc and Ocean stations this hypothesis reduces them all except two, and in the most important of all, viz. at Kaliana and Minicoy, it nearly annihilates them altogether when $m = 50$. The exceptions are Damargida, where there is a deficiency of matter indicated by -455, and Kalianpur, where there is an excess of matter indicated by $+338$. If the results of *horizontal* local attraction in the Table a few pages further on are examined, it will be seen there also, that the anomalies about Damargida there shown indicate a deficiency on each side of Damargida, and there is an excess south of Kalianpur between that place and Takul Khera. These tally, then, with those obtained by pendulum observations. They show that there are local irregularities in the crust, but that the general arrangement of the hypothesis here advanced removes or in a great measure reduces these apparent deviations from regularity, while the usual method does the contrary. Especially is the truth of the hypothesis supported by the fact, that it explains the singular circumstance that gravity can be greater at sea than on the coast.

It should be observed that in the construction of the Table the ellipticity has been taken $1 \div 295.2$. It is not difficult to show that for every additional unit in the denominator of the ellipticity the numbers in the last three columns of the Table in the 11 lines from left to right must be diminished by 0, 3, 8, 16, 25; 0, 1, 3, 3, 7; 0 respectively, arising from the correction for latitude.

It may also be remarked that Bangalore, Mangalore and Madras are nearly in the same latitude, so that their comparison is unaffected by the value of the ellipticity. The same is the case with Punnæ and Minicoy.

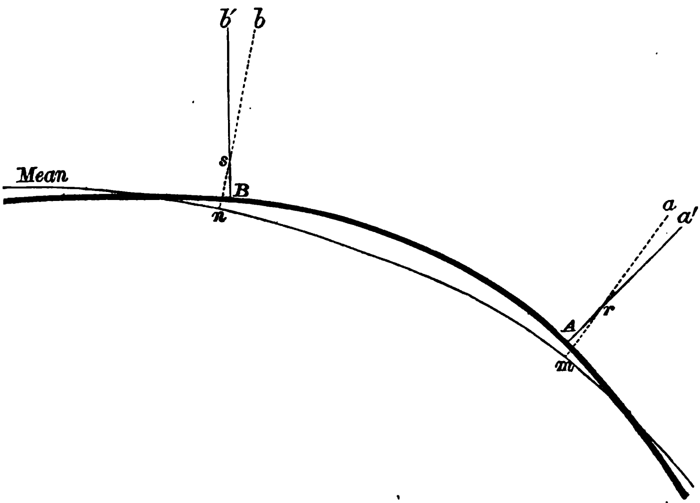
§ 2. *The form of separate parts of the surface.*

197. What has gone before leads to the determination of only the Mean Figure of the Earth. Our knowledge, however, of the surface—diversified as it is with mountains, plains, and oceans—is sufficient to show that particular parts of the surface depart from this mean figure.

We have shown already that the large effects of the Himalayas and the Ocean in India are very nearly compensated for by variations in density in the crust (Art. 191). The residual deflections, however, are not to be overlooked. It is to the consideration of these that we now call the attention of the student. In the course of our remarks some things will be explained which probably have not been so thoroughly understood in what has gone before as they will be now.

PROP. *To explain what is meant by the Sea-level, and to point out its use.*

198. In the diagram suppose A is the station from which we commence: and suppose the dark line AB to be the curve in which still water would lie, if a canal were cut from the sea



along the meridian through A northwards, and the sea were allowed to flow into it. This curve is called the SEA-LEVEL. Where the level changes owing to the ebb and flow of the tide the mean is taken.

The plumb-line at every place along this curve hangs at right angles to the curve at that place; because it is one condition of fluid equilibrium, that the resultant force at any point of the fluid surface acts in the normal at that point (Art. 108). This level-curve will partake, therefore, of all the irregularities of the plumb-line caused by local attraction. It indicates the general form of the surface, altered as it may be by the upheavings and sinkings which geology teaches us have most certainly taken place.

It is this curve which is meant when we speak of the Arc of Meridian, and it is the work of the Trigonometrical Survey to determine its form, and to measure the elevations and depressions of places on the meridian with reference to it. A and B are in fact points in which verticals through these places cut this level-curve, and are not necessarily the places themselves, which may be some feet above or below them. The exact contour of the earth's visible surface is obtained by finding the form of the level-curve or arc of meridian, and also the elevations or depressions of places, above or below this curve. The level-curve is not necessarily an ellipse: indeed most likely it is not one at all: but as it evidently does not differ much from a circle, short portions of it may be represented very well by an elliptic arc of small ellipticity.

The following calculation, adapted from the volume of the Ordnance Survey, will give some idea how much the level-curve may be affected in particular localities by irregularities in the mass below the surface.

PROP. *To find the effect of a mass not far below the surface of the earth on the sea-level or level-surface.*

199. Suppose the earth to be a sphere, since taking account of its ellipticity would introduce small quantities of the second order; a = radius, $h = a - k$, the depth of the centre of the attracting mass, taken to be more or less spherical; r the radius from the earth's centre to any point P of the sea-level;

θ the angle r makes with the radius through the centre of the attracting mass; v the ratio of the volumes of the attracting mass and a sphere, diameter 1 mile; g = gravity: take the excess of density of the mass above the density of that part = half the earth's mean density. Then the horizontal attraction of the mass on P

$$=\frac{gvk\sin\theta}{16a(a^2+k^2-2ka\cos\theta)^{\frac{3}{2}}}.$$

Tangent of angle of displacement of vertical at P

$$=\frac{vk\sin\theta}{16a(a^2+k^2-2ka\cos\theta)^{\frac{3}{2}}}=-\frac{1}{r}\frac{dr}{d\theta},$$

since the level-surface must be perpendicular to the displaced vertical;

$$\therefore \log_e\frac{r}{a}=\frac{v}{16a^2\sqrt{a^2+k^2-2ka\cos\theta}}+C;$$

$$\therefore r=a\left(1+\frac{v}{16a^2\sqrt{a^2+k^2-2ka\cos\theta}}+C\right);$$

because $v\div a^3$ is a very small fraction. The constant C must be found by the condition that the volume of the solid formed by the revolution of this curve about the diameter through the centre of the attracting mass must equal the volume of the sphere. As the curve can nowhere depart much from the sphere, it is obvious that $r-a$ the height of the curve surface above the sphere is of the order of $v\div a^3$, and therefore C must be of that order. The curve-surface will lie partly outside the sphere and partly inside. The condition is that its algebraic volume with reference to the sphere equals zero;

$$\therefore \int_0^\pi (r-a)\,2\pi a^2\sin\theta\,d\theta=0;$$

$$\therefore \frac{v}{16a^3}\sqrt{a^2+k^2-2ka\cos\theta}-Ck\cos\theta=0, \text{ between limits};$$

$$\therefore \frac{v}{16a^3}(a+k-a+k)+Ck(1+1)=0,\ C=-\frac{v}{16a^3};$$

$$\therefore r-a=\frac{v}{16a^2}\left(\frac{a}{\sqrt{a^2+k^2-2ka\cos\theta}}-1\right).$$

This is greatest immediately over the disturbing mass and then $=\frac{v}{16ha}$, h being small; call this H. As we know that the deflection of the vertical must always be insensible when θ is not small, we may make θ small. Then deflection

$$=\frac{vk\theta}{16a(a^2+k^2-2ka+ka\theta^2)^{\frac{3}{2}}}=\frac{v\theta}{16(h^2+a^2\theta^2)^{\frac{3}{2}}}$$

when $\theta=0$, deflection $=0$.

The maximum $\left(\text{when } \theta=\frac{h}{\sqrt{2}a}\right)$ is $\frac{v}{24\sqrt{3}\,h^2a}=D$ suppose. Let D be expressed in seconds and H in feet, h as before in miles; then

$$H=\frac{v}{12h},\quad D=\frac{9\sqrt{3}}{4\pi}\,\frac{v}{h^2}.$$

Thus if diameter of mass $=1$ mile, and depth $=0.5$ mile,

$$H=\frac{1}{6}=2 \text{ inches, while } D=\frac{9\sqrt{3}}{\pi}=4''.96.$$

If depth $=0.75$ mile, $H=1.33$ inch, $D=2''.20$.

This shows that a large observed amount of deflection may accompany an irregularity of the surface of the earth, the actual height of which is of an extremely minute order.

PROP. *To prove that the effect of a mass at the earth's surface, whether above or below, is to make the sea-level rise at any place through a space $V\div g$, where V is the potential of the mass for a point on the disturbed sea-level which is in the same vertical line with the place.*

200. Suppose a line drawn from the given place to the earth's centre, and θ the angle which a radius vector r to any point in the curve of the disturbed sea-level makes with that line. Then $\frac{-1}{r}\frac{dr}{d\theta}$ is the tangent of the angle between the radius r and the normal to the curve. This angle is the deflection caused by the horizontal attraction of the mass;

and its tangent equals the ratio of that attraction to gravity $= -\frac{1}{gr}\frac{dV}{d\theta}$, V being the potential of the mass for that point;

$$\therefore \frac{dr}{d\theta} = \frac{1}{g}\frac{dV}{d\theta};$$

$$\therefore r + \text{const.} = \frac{V}{g}.$$

Let $r = a$, where $V = 0$ or the horizontal attraction of the mass first becomes appreciable, and let V be the value of V at the place in question;

$$\therefore \text{rise of sea-level} = r - a = V \div g.$$

Ex. 1. The Himalayas may be considered, for problems of this kind, as a vast table-land about three miles high. By dividing it into vertical slices by vertical planes through a station on its surface, and making z the height through which the sea-level immediately below the station is raised, it is not difficult to show by the Integral Calculus, that the potential of the lower part of the table-land, that between the true sea-level and a level parallel to it through that point on the disturbed sea-level which is under the station, for that point

$$= \Sigma . \frac{2\pi\rho}{i}\left(uz - \frac{z^2}{2}\right),$$

u being the horizontal length of any one of the slices, and i the number of slices.

So for the upper part of the table-land the potential

$$= \Sigma . \frac{2\pi\rho}{i}\left\{u(h - z) - \frac{(h-z)^2}{2}\right\},$$

h being the height of the table-land above the undisturbed sea-level. Adding these together

$$V = \Sigma . \frac{2\pi\rho}{i}\left\{uh - \frac{z^2 + (h-z)^2}{2}\right\}$$

$$= 2\pi\rho\left\{h\frac{\Sigma . u}{i} - \frac{z^2 + (h-z)^2}{2}\right\}.$$

Let d be the length of the mean horizontal diameter of the table-land through the station: then

$$V = \frac{3g}{4c}\left\{h\frac{d}{2} - \frac{z^2 + (h-z)^2}{2}\right\}.$$

Substituting for V from the Proposition,

$$z = \frac{3hd}{8c},$$

neglecting the second term, which will be evanescent.

Ex. 2. Suppose that below the Himalayas there is a uniform attenuation of the crust reaching down through a depth m times the height of the table-land, and equal in amount to the mass of the table-land. We will find the rise of the sea-level under these circumstances. The effect of this attenuation below the sea-level is to make the sea-level rise still more than in the last example. Putting z', measured upwards, for the rise thus caused, the potential of the attenuation equals the difference of the potentials of two such attenuations of mass reaching down through depths z' and $z' + mh$

$$= \Sigma \frac{2\pi\rho}{mi}\left[\left\{u(z' + mh) - \frac{(z'+mh)^2}{2}\right\} - \left(uz' - \frac{z'^2}{2}\right)\right]$$

$$= \frac{3g}{4c}\left(h\frac{\Sigma.u}{i} - \frac{2hz' + mh^2}{2}\right) = \frac{3g}{4c}\left(\frac{hd}{2} - hz' - \frac{m}{2}h^2\right)$$

$$= \frac{3g}{4c}\frac{hd}{2}\left(1 - \frac{mh}{d}\right) \text{ very nearly.}$$

Hence the rise of the sea-level is as much again, under the hypothesis of attenuation, as it is without it, when mh is small (it is only 150 or 300 in cases we consider) compared with d.

201. This latter example has been worked out with reference to the extension of the Pendulum Observations, alluded to in Art. 196, to the lofty plateau of the Himalayas. The late Captain Basevi, R.E.—whose untimely death this year in the midst of his observations at that height science may well deplore—completed one set of observations at a

station called More, latitude 33° 16′, longitude 77° 54′, height 15,500 feet; and Colonel J. T. Walker, R.E., Superintendent of the Great Trigonometrical Survey of India, has obligingly forwarded to the author, while these sheets are passing through the press, the following results:

Number of vibrations of a seconds pendulum
in 24 hours at More............................85978.69,
ditto at Punnæ (Cape Comorin)........85978.18,

after correcting, in both cases, for barometer and thermometer, but not for height and latitude.

These results are practically the same: and therefore the amount of gravity at the two stations is the same.

The latitude of Punnæ is 8° 10′, its height above the sea 44 feet, which may be neglected. Hence by Clairaut's theorem (Art. 122) gravity at More is greater than at Punnæ, owing to latitude, by

$$\frac{1}{2}\left(\frac{5}{2}m-\epsilon\right)(\cos 16^\circ 20' - \cos 66^\circ 32') = 0{\cdot}0014770g,$$

since $m=\dfrac{1}{289}$ and $\epsilon=\dfrac{1}{295.2}$.

On the other hand gravity at More is less than at Punnæ, owing to its superior height. After this and the effect of latitude are allowed for, the residuum difference between gravity at More and at Punnæ forms the basis of a further test of the theory regarding the constitution of the earth's crust referred to in Art. 196 and previous Articles.

If h be the height of the plateau above the sea-level ascertained from the survey operations, then its height from the sea-level undisturbed by this local effect is, by the two examples given in the last Article,

$$h+\frac{3hd}{4c}\left(1-\frac{mh}{d}\right),$$

and the deduction for height

$$=\frac{2}{c}\left\{h+\frac{3hd}{4c}\left(1-\frac{mh}{d}\right)\right\}g.$$

Also, by Art. 79, the Resultant Vertical Attraction of the plateau

$$=\frac{3g(1+m)^2}{4cd}\left\{h+\frac{3hd}{4c}\left(1-\frac{mh}{d}\right)\right\}^2.$$

The surface of the plateau may be regarded as a long ellipse, 300 miles across and 1500 miles long; equal to the area of a circle of diameter 670 miles. Put in the formulæ $d=670$, $h=15500$ feet $=2.936$ miles, $m=50$, $c=3956$. Then

$$\text{corrected height}=3.224 \text{ miles},$$

$$\text{correction of gravity for height}=0{\cdot}0016299g.$$

Hence gravity at More when corrected for latitude and height is greater than at Punnæ by $0.0001529g$. The formula for correction for local attraction gives

$$\text{correction for local attraction}=0.0001483g,$$

which is so very near the residuum error just found as to bear strong testimony to the truth of the theory here advocated.

Prop. *To explain what is meant by Astronomical and Mean Amplitudes.*

202. Let AB be the geodetic arc measured along the sea-level or reduced to that level. This arc may lie above or below the mean ellipse of the whole earth or upon it. In the diagram (Art. 198) we suppose it to lie above the mean ellipse. From A and B draw perpendiculars (not shown in the diagram) upon the mean ellipse; and take the points m and n on the mean ellipse at equal distances on opposite sides of the perpendiculars, so that the length mn measured along the mean ellipse may equal the length AB of the geodetic arc: and at m and n draw the dotted normals ma and nb to the mean ellipse. Then the geodetic arc is properly represented (Art. 162, 6) by mn on the mean ellipse, and the angle between ma and nb is the Mean Amplitude of the measured arc. It may be thought that, as the curvature of AB may differ slightly from that of mn, the length mn may be too great or too small fairly to represent the geodetic arc, if it be made exactly equal to that of AB.

But this is not the case. The two arcs may not be exactly parallel to each other; every element of the one may be slightly inclined to the corresponding element of the other: but this would introduce quantities only of the second order, which we neglect. Also, although owing to the convergence of the normals from A and B inwards, the length AB will be greater or less than the distance of the normals along the mean ellipse, as AB is above or below that ellipse, yet in a whole quadrant there will be a complete compensation, and the length of the quadrant of the mean ellipse will equal that of the level curve, and may therefore be divided into portions exactly equal to the geodetic arcs forming the level-curve quadrant. The mean amplitude is found from the geodetic arc, and the mean axes (obtained in Art. 173) by means of the formula of Art. 150.

Let Aa' and Bb' (crossing the others in r and s), be the lines in which the plumb-lines at A and B hang. The angle which they include is the observed or Astronomical Amplitude of the arc, because it is measured by the corresponding arc in the heavens, defined by the points in which the plumb-line at its extremities intersects the celestial vault. In the diagram, the astronomical exceeds the mean amplitude by the sum of the angles ara', bsb', or, algebraically speaking, by their difference, for if one of them is positive, as drawn in the diagram, the other is negative.

203. The mean amplitude differs from the astronomical amplitude solely because of the irregularities which cause the geodetic arc to differ in position from the mean arc. Hence the difference between the mean amplitude and the astronomical amplitude measures exactly the difference of meridian deflection caused by local attraction at the two extremities of the arc. In fact, if, in the diagram Art. 198, we imagine the earth to settle down into its mean spheroid, the points A and B would fall on m and n, the plumb-line at m and n would hang in the normals, and therefore ara' and bsb' exactly equal the deflections caused by local attraction. In the diagram, as these angles are drawn, one is negative and one is positive, and therefore their algebraical difference is, in this particular case, their actual sum.

The following Prop. will show that the Mean Amplitude is to be found as above, viz. from the geodetic arc and the mean axes, even if the part of the mean ellipse selected by Bessel's method (Art. 166) is used to represent the geodetic arc.

PROP. *To prove that the lengths of two elliptic arcs (axes parallel to each other) between two places, as much as twelve degrees and a half apart, differ by an insensible quantity; their ellipticities being small, their difference not exceeding the earth's mean ellipticity.*

204. Let s be the length of an elliptic arc between the stations, l and l' the observed latitudes of the extremities, λ and m the amplitude and middle latitude. Let c be the chord, r and θ, r' and θ' the polar co-ordinates from the centre of the ellipse to the extremities of the arc, a and b the semiaxes;

$$\therefore\ c^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta') = 2rr'\{1-\cos(\theta-\theta')\} + (r-r')^2,$$

$$r = a(1-\epsilon\sin^2 l),\quad r' = a(1-\epsilon\sin^2 l').$$

Also $\quad \tan\theta = (1-2\epsilon)\tan l,\quad \theta = l - \epsilon\sin 2l;$

$$\therefore\ \theta-\theta' = \lambda - 2\epsilon\sin\lambda\cos 2m;$$

$$\therefore\ 1-\cos(\theta-\theta') = 1-\cos\lambda - 2\epsilon\sin^2\lambda\cos 2m$$

$$= 2\sin^2\frac{1}{2}\lambda\,\{1-2\epsilon(1+\cos\lambda)\cos 2m\};$$

$$\therefore\ c^2 = 4a^2\sin^2\frac{1}{2}\lambda\,\{1-2\epsilon(1+\cos\lambda)\cos 2m - \epsilon(\sin^2 l+\sin^2 l')\}$$

$$= 4a^2\sin^2\frac{1}{2}\lambda\,[1-\epsilon\{1+(2+\cos\lambda)\cos 2m\}];$$

$$\therefore\ \sin\frac{1}{2}\lambda = \frac{c}{2a}\left[1+\frac{1}{2}\epsilon\{1+(2+\cos\lambda)\cos 2m\}\right];$$

$$\therefore\ \frac{\lambda}{2} = \sin^{-1}\frac{c}{2a} + \frac{1}{2}\epsilon\{1+(2+\cos\lambda)\cos 2m\}\frac{c}{\sqrt{4a^2-c^2}}$$

$$= \sin^{-1}\frac{c}{2a} + \frac{1}{2}\epsilon\{1+(2+\cos\lambda)\cos 2m\}\tan\frac{1}{2}\lambda.$$

Now $s = a\left(1 - \frac{1}{2}\epsilon\right)\lambda - \frac{3}{2}a\epsilon\sin\lambda\cos 2m$, by Art. 150;

$$\therefore\ s = a(2-\epsilon)\sin^{-1}\frac{c}{2a} + a\epsilon\{1 + (2+\cos\lambda)\cos 2m\}\tan\frac{1}{2}\lambda$$
$$-\frac{3}{2}a\epsilon\sin\lambda\cos 2m$$

$$= (a+b)\sin^{-1}\frac{c}{2a} + (a-b)\left\{1 + \frac{1}{2}(1-\cos\lambda)\cos 2m\right\}\tan\frac{1}{2}\lambda.$$

Taking the variations of s with respect to a and b, c being constant, as also λ and m because they occur in small terms, we have the difference in length of two arcs joining the stations and belonging to different ellipses, only having their axes parallel.

$$\therefore\ \delta s = (\delta a + \delta b)\sin^{-1}\frac{c}{2a} - \frac{a+b}{a}\,\frac{c\,\delta a}{\sqrt{4a^2 - c^2}}$$

$$+ (\delta a - \delta b)\left\{1 + \frac{1}{2}(1-\cos\lambda)\cos 2m\right\}\tan\frac{1}{2}\lambda.$$

The terms being small we may approximate;

$$\therefore\ \delta s = (\delta a + \delta b)\frac{1}{2}\lambda - 2\tan\frac{1}{2}\lambda\,.\,\delta a$$

$$+ (\delta a - \delta b)\left\{1 + \frac{1}{2}(1-\cos\lambda)\cos 2m\right\}\tan\frac{1}{2}\lambda$$

$$= (\delta a + \delta b)\left(\frac{1}{2}\lambda - \tan\frac{1}{2}\lambda\right) + (\delta a - \delta b)\frac{1}{2}\tan\frac{1}{2}\lambda(1-\cos\lambda)\cos 2m$$

$$= (\delta a + \delta b)P + (\delta a - \delta b)Q\cos 2m, \text{ suppose,}$$

$$= (P + Q\cos 2m)\delta a + (P - Q\cos 2m)\delta b;$$

δa and δb are two arbitrary increments of a and b. We will find the least values of these which will produce a given increase δs to the arc: that is, the values which make $\delta a^2 + \delta b^2$ a minimum.

$$\therefore \ \delta a^2 + \left\{\frac{\delta s - (P + Q\cos 2m)\,\delta a}{P - Q\cos 2m}\right\}^2 = \text{a minimum};$$

$$\therefore \ \{(P - Q\cos 2m)^2 + (P + Q\cos 2m)^2\}\,\delta a$$
$$= (P + Q\cos 2m)\,\delta s;$$

$$\therefore \ \delta a = \frac{P + Q\cos 2m}{P^2 + Q^2\cos^2 2m}\,\frac{\delta s}{2} \text{ and } \delta b = \frac{P - Q\cos 2m}{P^2 + Q^2\cos^2 2m}\,\frac{\delta s}{2},$$

$$\delta a^2 + \delta b^2 = \frac{1}{P^2 + Q^2\cos^2 2m}\,\frac{\delta s^2}{2}.$$

This is least when $m = 0$ or 90°; then

$$\delta a = \frac{P \pm Q}{P^1 + Q^2}\,\frac{\delta s}{2}, \qquad \delta b = \frac{P \mp Q}{P^2 + Q^2}\,\frac{\delta s}{2};$$

$$\text{and} \quad \delta a \sim \delta b = \frac{Q\delta s}{P^2 + Q^2}.$$

Let one of the ellipses be equal to the ellipse of the earth's mean figure, a and b being the semi-axes; then δa and δb will be the excess (or defect, if negative) of the semi-axes of the other ellipse: this latter ellipse is taken to be the ellipse which most nearly coincides with the actual arc s of the level curve and therefore represents it. The arc of the mean ellipse which corresponds with s of the actual measured arc will not necessarily have precisely the same middle latitude, although the chord c is of the same length. But as the middle latitude will differ only by a quantity of the order of the ellipticity this difference will not appear in the result because we neglect the square of the ellipticity.

We will put $\delta s = \text{arc } 1'' = 0{\cdot}0193$ mile, 1° being $69{\cdot}5$ miles: and will find the value of λ which will make $\delta a \sim \delta b$ as large as the whole compression of the earth's pole, viz. 13 miles. This gives

$$P^2 + Q^2 \div Q = 0.0193 \div 13 = 0.0015,$$

$$\text{or } \left(\frac{\lambda}{2} - \tan\frac{\lambda}{2}\right)^2 + \frac{1}{4}\tan^2\frac{\lambda}{2}\,(1 - \cos\lambda)^2$$

$$= 0.00075 \tan\frac{\lambda}{2}\,(1 - \cos\lambda).$$

A slight inspection of this equation shows that λ must be small. Expand in powers of λ; then

$$\left(\frac{1}{9}+1\right)\left(\frac{\lambda}{2}\right)^3=0.0015, \quad \text{or} \quad \left(\frac{\lambda}{2}\right)^3=0.00135;$$

$$\therefore\ \lambda=0.22 \text{ (in arc)}=0.22\times 57^\circ.3 \text{ (in degrees)}=12^\circ.6.$$

This shows, that in an arc of meridian as much as twelve degrees and a half in length it would require a departure from the mean ellipse equal to the whole actual compression of the pole of the earth in order to produce so slight a difference in the length as 1″. Hence we may conclude that the difference in length between the mean arc (according even to Bessel's way of representing it) and the actual arc is in fact an insensible quantity, since an extravagant hypothesis regarding the departure of the form of the arc in question from the mean form will not produce a difference of length of more than 1″. The property here proved shows, then, that in Art. 202 mn was properly made equal in length to AB, that it might represent AB on the mean ellipse.

PROP. *To estimate the relative amount of local attraction in the plane of the meridian at stations on the Indian Arc.*

205. We take the following data from the volume of the British Ordnance Survey, p. 757. From these we calculate the mean amplitudes by means of the formula derived from Art. 151, viz.

$$\lambda=\frac{2s}{a+b}\left(1+\frac{3}{2}\epsilon\cos 2m\right),$$

a, b, and ϵ are 20926184, 20855304, $\frac{1}{295.2}$ (Art. 173).

	Punnæ to Putchapolliam.	Putchapolliam to Dodagoontah.	Dodagoontah to Namthabad.	Namthabad to Damargida.	Damargida to Takul Khera.	Takul Khera to Kalianpur.	Kalianpur to Kaliana.
$2m=$	19° 9′ 13″.408	23° 59′ 34″.441	28° 5′ 45″.727	33° 9′ 8″.854	39° 9′ 6″.824	45° 13′ 2″.794	53° 37′ 59″.584
$2s=$	2058347.4	1454772.6	1523626.8	2146821.8	2211079.6	2194729.8	3922276.0
Ast. $\lambda=$	2° 50′ 11″.144	2° 0′ 9″.889	2° 6′ 1″.397	2° 57′ 21″.730	3° 2′ 36″.240	3° 1′ 19″.730	5° 23′ 37″.060
$\log \frac{3}{2}=$	0.1760913	0.1760913	0.1760913	0.1760913	0.1760913	0.1760913	0.1760913
$\log \epsilon=$	$\bar{3}.5298837$	$\bar{3}.5298837$	$\bar{3}.5298837$	$\bar{3}.5298837$	$\bar{3}.5298837$	$\bar{3}.5298837$	$\bar{3}.5298837$
$\log \cos 2m=$	$\bar{1}.9752671$	$\bar{1}.9607541$	$\bar{1}.9455471$	$\bar{1}.9228387$	$\bar{1}.8895678$	$\bar{1}.8478306$	$\bar{1}.7730197$
	$\bar{3}.6812421$	$\bar{3}.6667291$	$\bar{3}.6515221$	$\bar{3}.6288137$	$\bar{3}.5955428$	$\bar{3}.5538056$	$\bar{3}.4789947$
$1+\frac{3}{2}\epsilon \cos 2m=$	1.0048000	1.0046423	1.0044825	1.0042542	1.0039404	1.0035794	1.0030130
$\log (1+\frac{3}{2}\epsilon \cos 2m)=$	0.0020796	0.0020115	0.0019424	0.0018437	0.0017080	0.0015318	0.0013065
$\log 2s=$	6.3135187	6.1627952	6.1828786	6.3317960	6.3446044	6.3413811	6.5935382
$\log (a+b)=$	7.6209839	7.6209839	7.6209839	7.6209839	7.6209839	7.6209839	7.6209839
$\log \lambda=$	$\bar{2}.6946144$	$\bar{2}.5438228$	$\bar{2}.5638371$	$\bar{2}.7126558$	$\bar{2}.7253285$	$\bar{2}.7219490$	$\bar{2}.9738608$
$\log \sin 1''=$	$\bar{6}.6855749$	$\bar{6}.6855749$	$\bar{6}.6855749$	$\bar{6}.6855749$	$\bar{6}.6855749$	6.6855749	$\bar{6}.6855749$
$\log \lambda''=$	4.0090395	3.8582479	3.8782622	4.0270809	4.0397536	4.0363741	4.2882859
Mean $\lambda=$	10210″.32	7215″.19	7555″.48	10643″.41	10958″.56	10873″.62	19421″.65
or	2°50′10″.32	2° 0′ 15″.19	2° 5′ 55″.48	2° 57′ 23″.41	3° 2′ 38″.56	3° 1′ 13″.62	5° 23′ 41″.65
Mean—Ast. Amp. $=$	− 0″.82	+ 5″.30	− 5″.92	+ 1″.68	+ 2″.32	− 6″.11	+ 4″.59

Northern deflections of Plumb-line relatively to Punnæ, and their equivalents in terms of gravity.

Punnæ.	Putchapolliam.	Dodagoontah.	Namthabad.	Damargida.	Takul Khera.	Kalianpur.	Kaliana.
0	− 0″.82	+ 4″.48	− 1″.44	+ 0″.24	+ 2″.56	− 3″.55	+ 1″.04
0	− 0.0000040 g.	+ 0.0000217 g.	− 0.0000070 g.	+ 0.0000012 g.	+ 0.0000124 g.	− 0.0000172 g.	+ 0.0000050 g.

Stations.	*Latitudes.*	*Amplitudes.*	*Distances in feet.*
Punnæ	8° 9′ 31″.132	− 9° 53′ 44″.160	− 3591784.3
Putchapolliam	10° 59′ 42″.276	− 7° 3′ 33″.016	− 2562610.6
Dodagoontah	12° 59′ 52″.165	− 5° 3′ 23″.127	− 1835224.3
Namthabad	15° 5′ 53″.562	− 2° 57′ 21″.730	− 1073410.9
Damargida	18° 3′ 15″.292	...	...
Takul Khera	21° 5′ 51″.532	3° 2′ 36″.240	1105539.8
Kalianpur	24° 7′ 11″.262	6° 3′ 55″.970	2202904.7
Kaliana	29° 30′ 48″.322	11° 27′ 33″.030	4164042.7

The results are given in the table on the opposite page. As the deflections are very small quantities and are the differences of very much larger ones, no doubt they would be somewhat modified if the *square* of the ellipticity were not neglected. But they are sufficiently near the truth for our present purpose. They will not be sensibly affected by changing the ellipticity which we have taken $1 \div 295.2$. For, if we consider the mean radius or half $(a+b)$ to be absolutely constant, the formula for λ makes the mean λ only about 1-60,000th smaller for each additional unit in the denominator of the ellipticity.

The angles are not the *absolute* deflections, but only the relative deflections at one station compared with another. But from Art. 191 it appears to be most likely, that the absolute meridian deflection at Damargida is only 0″.05, which is an evanescent quantity. Taking, therefore, the angles with reference to Damargida, we have the actual northernly deflections at Punnæ −0″.24, Putchapolliam −1″.04, Dodagoontah + 4″.25, Namthabad − 1″.68, at Damargida 0″.00, Takul Khera + 2″.32, Kalianpur − 3″.79, Kaliana + 0″.80. These are all small: the largest of them is that at Kalianpur.

206. The quantities above deduced are independent of any theory regarding the structure of the earth's mass. We may, however, endeavour to trace these resulting effects to their causes. In a former part of this treatise (Art. 89) it has been explained that two visible causes exist producing deflection, viz. the mountain mass on the north of India and the vast ocean on the south. It has also been shown (Art. 94) that a hidden cause of deflection may lie below, in the variation of the density of the earth's crust. The effect of the two

visible causes has been estimated approximately by the author for four of the above stations as follows (*Phil. Trans.* 1859):

Deflections northwards at	Punnæ,	Damargida,	Kalianpur,	Kaliana.
Caused by the Mountains...	2″.50*?	6″.79	12″.05	27″.98
Caused by the Ocean.........	19″.71	10″.44	9″.00	6″.18
Totals	22″.21	17″.23	21″.05	34″.16

By these quantities the latitudes are diminished. Therefore the excess of the mean over the astronomical amplitudes =

$$-4''.98, \quad \div 3''.82, \quad +13''.11.$$

These differ considerably from the differences of amplitude deduced from the arcs in the last Article. This shows us that there must be irregularities in the density of the crust below: their effect on the amplitudes is shown as follows:

Differences of amplitude determined in last Article	+0″.24, −3″.79, + 4″.59,
Effects of mountains and ocean	−4 .98, +3 .82, +13 .11,
Consequent effect of the hidden causes in the crust below	+5 .22, −7 .61, − 8 .52.

The hidden cause increases the astronomical amplitudes of the northern and middle of these three divisions of the Great Indian Arc, that is, makes the plumb-lines hang at a greater angle to each other; and diminishes the amplitude of the southern division, or makes the plumb-lines at its extremities hang less inclined to each other. This seems to imply that somewhere between Kaliana and Damargida there is an excess of density, and between Damargida and Punnæ there is a deficiency. By referring to the table, last two lines, it appears that the excess must be near Kalianpur and the defect near Damargida. For the deflection at Kalianpur is considerably south, and at Takul Khera north: and on both sides of Damargida the deflection is from that station. See p. 208.

* This amount was not calculated in the Paper in the *Philosophical Transactions* alluded to above, as it was not there required. It has been since roughly obtained, in the same manner as the others, for the present purpose.

PROP. *To prove that the length of a mean arc of longitude (according to Bessel) is sensibly the same as the geodetically measured arc, if it do not exceed fifteen degrees in length.*

207. Let S be the length of the arc, l the latitude, L the longitudinal amplitude (i.e. the difference of the longitudes of its two extremities), c the chord. Then by Art. 153,

$$S = L \cos l \{a + (a - b) \sin^2 l\},$$

$$c = 2 \cos l \{a + (a - b) \sin^2 l\} \sin \frac{1}{2} L.$$

When a and b vary, c and l remain constant, but S and L vary. Hence

$$\delta S = \delta L \cos l \{a + (a - b) \sin^2 l\} + L \cos l \{\delta a + (\delta a - \delta b) \sin^2 l\},$$

$$0 = \{a + (a - b) \sin^2 l\} \cos \frac{1}{2} L \delta L$$

$$+ 2 \{\delta a + (\delta a - \delta b) \sin^2 l\} \sin \frac{1}{2} L;$$

$$\therefore \ \delta S = \left(L - 2 \tan \frac{1}{2} L\right) \cos l \{\delta a + (\delta a - \delta b) \sin^2 l\};$$

$$\therefore \ \delta a + (\delta a - \delta b) \sin^2 l = \frac{\delta S}{\left(L - 2 \tan \frac{1}{2} L\right) \cos l} = n, \text{ suppose:}$$

δa and δb are arbitrary increments of a and b and produce the increment δS in the arc of longitude. We will find the least values of δa and δb, or those which make $\delta a^2 + \delta b^2$ a minimum;

$$\therefore \ \sin^4 l \delta a^2 + \{(1 + \sin^2 l) \delta a - n\}^2 = \text{a minimum};$$

$$\therefore \ \{\sin^4 l + (1 + \sin^2 l)^2\} \delta a = n (1 + \sin^2 l);$$

$$\therefore \ \delta a = \frac{(1 + \sin^2 l) n}{\sin^4 l + (1 + \sin^2 l)^2}, \quad \delta b = -\frac{\sin^2 l \,.\, n}{\sin^4 l + (1 + \sin^2 l)^2};$$

$$\therefore\ \delta a^2 + \delta b^2 = \frac{n^2}{\sin^4 l + (1+\sin^2 l)^2}$$

$$= \frac{\delta S^2}{\cos^2 l\,\{\sin^4 l + (1+\sin^2 l)^2\}\left\{L - 2\tan\frac{1}{2}L\right\}^2}.$$

This is least when $l = 0$; then

$$\delta a = n,\ \ \delta b = 0,\ \ \delta a - \delta b = n = \frac{\delta S}{L - 2\tan\frac{1}{2}L}.$$

Now put $\delta a \sim \delta b = 13$ miles, $\delta S =$ arc $1''$ of a great circle $= 0.0193$ mile;

$$\therefore\ L - 2\tan\frac{1}{2}L = 0.0193 \div 13 = 0.0015.$$

This shows that L must be small. Expanding we have

$$L^3 = 0.018,\ \ L = 0.262 \text{ (in arc)} = 0.262 \times 57^\circ.3 = 15^\circ.$$

This shows that in an arc of longitude as much as fifteen degrees long (the length in miles depending, of course, on the latitude) it would require a departure from the mean ellipse equal to the whole actual compression of the pole of the earth to produce a difference in the length of the arc of only 0.0193 mile, or 102 feet. If it require so extravagant an hypothesis regarding the departure of the form of the arc from the mean form to produce so small a difference in the length, we may conclude that the actual difference in length of the actual arc and the mean arc of longitude is insensible, if the arc be no longer than fifteen degrees.

PROP. *To explain what effect local attraction will have upon the mapping of a country.*

208. The distances of places on the earth's surface referred to the mean spheroid are known from the measured arcs (see Art. 202); and by the formulæ in Art. 150, and the mean axes, the differences of latitude and longitude can

be accurately determined, and the places laid down accordingly in a map would have their relative positions correctly assigned. But we have no direct means of ascertaining these distances. This is the case also if Bessel's method is used (Art. 166). For in Arts. 202, 207 it has been shown that the actual lengths of arcs measured by the Survey differ from the lengths of the mean arcs (measured by this method) by inappreciable quantities, if the arcs are not chosen inordinately long, a thing which is never done. These measured arcs may therefore be used in this calculation instead of the mean arcs; and this convenient result is arrived at, that the relative position of places laid down on a map as determined by the Survey operations is not sensibly affected by any deviations of the form of the surface from the mean form, caused by those upheavings and depressions which geology shows us have undoubtedly taken place. The position of the map itself on the mean terrestrial spheroid would be fixed by ascertaining the absolute latitude and longitude of some one place in it. These would, of course, be affected by local attraction in both methods.

It thus appears that a map constructed wholly from geodetical measurements will be accurate in itself, that is, the relative position of places marked down in it will be correct. But the map itself will be as much out of its place on the terrestrial spheroid as the latitude and longitude of the station which fixes the map are erroneous in consequence of local attraction at that place. Also if any place is afterwards inserted in the map by observations made upon the heavens, the place will be out of its proper position in the map by the difference in deflection of the plumb-line at that place and at the place the latitude and longitude of which fix the map.

PROP. *To estimate the degree of departure of an arc of meridian between two stations from the curvature of the mean arc.*

209. Suppose an ellipse, with one axis parallel to the earth's axis, to be drawn through the extremities of the arc and nearly coinciding with the arc and representing it. Let the origin of co-ordinates be very near the centre of this

ellipse; r and θ, r' and θ' polar co-ordinates to the extremities of the arc from the centre of the ellipse; α and β rectangular co-ordinates to the centre of the ellipse, and therefore very small quantities. Hence the equation to this ellipse is

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1,$$

$$\therefore\ x^2+y^2 \text{ or } r^2=a^2+2\alpha x+2\beta y-2\epsilon(a^2-x^2)$$
$$=a^2+2a\alpha\cos\theta+2a\beta\sin\theta-2a^2\epsilon\sin^2\theta;$$
$$\therefore\ r=a+\alpha\cos\theta+\beta\sin\theta-a\epsilon\sin^2\theta.$$

Let R, C, C' be the values of r at the mid-latitude and at the extremities of the arc;

$$\therefore\ R=a+\alpha\cos m+\beta\sin m-(a-b)\sin^2 m,$$
$$C=a+\alpha\cos l+\beta\sin l-(a-b)\sin^2 l,$$
$$C'=a+\alpha\cos l'+\beta\sin l'-(a-b)\sin^2 l'.$$

To eliminate α and β multiply by 1, M, and N; add, and make the coefficients of α and β vanish;

$$\therefore\ \cos m+M\cos l+N\cos l'=0,\ \ \sin m+M\sin l+N\sin l'=0;$$

$$\therefore\ N=-\frac{\sin(m-l)}{\sin(l'-l)}=-\frac{1}{2}\sec\frac{1}{2}\lambda=M;$$

$$R+MC+NC'$$
$$=a(1+M+N)-(a-b)(\sin^2 m+M\sin^2 l+N\sin^2 l')$$
$$=a(1+2M)-\frac{1}{2}(a-b)\{1-\cos 2m+2M(1-\cos\lambda\cos 2m)\}$$
$$=\frac{1}{2}(a+b)(1+2M)+\frac{1}{2}(a-b)(1+2M\cos\lambda)\cos 2m$$
$$=\frac{1}{2}(a+b)\left(1-\sec\frac{1}{2}\lambda\right)+\frac{1}{2}(a-b)\left(1-\sec\frac{1}{2}\lambda\cos\lambda\right)\cos 2m.$$

Let δa and δb be the excess of the semi-axes of the actual arc above the axes of an ellipse equal to the mean ellipse and

passing through the extremities of the arc, the axes of the two ellipses being parallel. Then taking the variations, the distance required, or δR,

$$= \frac{\delta a + \delta b}{2}\left(1 - \sec\frac{\lambda}{2}\right) + \frac{\delta a - \delta b}{2}\left(1 - \sec\frac{\lambda}{2}\cos\lambda\right)\cos 2m$$

$$= -\frac{\lambda^2}{16}(\delta a + \delta b) + \frac{3\lambda^2}{16}(\delta a - \delta b)\cos 2m, \text{ neglecting } \lambda^4\ldots$$

210. Ex. Let the arc be that between Kaliana (29° 30′ 48″) and Damargida (18° 3′ 13″): and let it be supposed to be part of the ellipse deduced in Art. 155, Ex. 2.

In this case $\delta a = 56955$, $\delta b = 19707$ (see Art. 173);

$$\therefore\ \lambda = 11^\circ\,27'\,33'' = 0.2 \text{ in arc}, \quad \cos 2m = \cos 47^\circ\,34' = 0.6747;$$

$$\therefore\ \delta R = -0.0025\,(\delta a + \delta b) + 0.0050\,(\delta a - \delta b)$$

$$= 0.0025\,\delta a - 0.0075\,\delta b = -5 \text{ feet}.$$

Although the ellipse compared with the mean ellipse differs much in the length of its axes, yet its depression at the middle point of an arc eleven degrees long, is only 5 feet.

PROP. *Geodesy furnishes at present no evidence, in proof or disproof, of the upheaval or depression of the Earth's surface as suggested by geological phenomena.*

211. It might at first seem from the last Article that geodesy proves, that the position of the arc has not been sensibly changed, and that geological processes have not affected it. But it must be observed, that the comparison of the arc has been made not with the mean ellipse itself, but with an ellipse equal in dimensions to the mean ellipse and with axes parallel (because the latitudes are measured in all the ellipses from the same or parallel lines). This ellipse was so drawn as to pass through the extremities of the arc; but we have no means of knowing that the mean ellipse itself passes through those two points: it may lie above them or below them. We have no means of ascertaining the precise

position of the centre of the mean ellipse. The only way of doing this is to make a geodetic measurement of the whole of one meridian from pole to pole. Till this is done we have no evidence of any particular arc lying above or below the mean, i.e. of its having been elevated or depressed. The greatest geological changes of level, therefore, are perfectly consistent with all we know by geodesy of the surface of the Earth.

212. It has been explained, that in consequence of the inequalities of the Earth's surface the observations, whether made on the pendulum or in geodetic operations, are all referred to the SEA-LEVEL; that is, to that surface which the sea would form if allowed to percolate by canals through the continents. The sea is thus taken as the basis of our measurements; and is generally assumed to have a spheroidal form. But it is possible that these local disturbing forces, arising from attraction, may have the effect of crowding up the waters in the direction in which the forces act, so as sensibly to alter the sea-level from the spheroidal form. This we shall proceed to examine.

PROP. *To find the effect of a small disturbing force in changing the Level of the Sea.*

213. Let U be the disturbing force and du an element of the line u along which it acts. Then $\int U du$ must be added to the potential in the equation of fluid equilibrium of Art. 108.

$$\therefore \int \frac{dp}{\rho} = V + \frac{w^2}{2} r^2 (1 - \mu^2) + \int U du = \text{const. at the surface.}$$

Putting $w^2 = m . \dfrac{E}{a^3}$ and substituting for V from Art. 121,

$$\text{constant} = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right) \frac{Ea^2}{r^3} \left(\frac{1}{3} - \mu^2\right) + \frac{m}{2} \frac{E}{a^3} (1 - \mu^2) + \int U du.$$

When the small force U is neglected, $a \div r = 1 + \epsilon . \mu^2$, by the equation to the ellipse. Hence, neglecting small quan-

tities of the second order, dividing by E, multiplying by a, and transposing, the above equation must become

$$\frac{\mathrm{a}}{r} = 1 + \epsilon . \mu^2 - \frac{\mathrm{a}}{E}\int U du.$$

Now $\frac{1}{r}\frac{dr}{d\theta}$ is the tangent of the angle between r and the normal, $= \tan\psi$ suppose: and the angle through which the normal is thrown back by the force U

$$= \delta\psi = \delta . \tan\psi = -\delta . r\frac{d}{d\theta}\frac{1}{r} = \frac{\mathrm{a}}{E}U\frac{du}{d\theta}.$$

Hence the element ds of the undisturbed meridian line on the surface of the sea is elevated, on the side towards which U acts, by the space

$$ds . d\psi = \frac{\mathrm{a}}{E}U\frac{du}{d\theta}ds = \frac{\mathrm{a}^2}{E}U du = \frac{U}{g}du;$$

$$\therefore \text{ whole elevation of the sea-level} = \frac{1}{g}\int U du,$$

integrated between the limits.

This will be true of any small force acting in any direction. But as the surface of the ocean is nearly spherical, it is only the horizontal part of the force which will have any sensible effect. For suppose that R is the part of the force resolved vertically, which we may consider to be in the direction of r, the radius vector, tending to shorten it. Then for Udu we must put $-Rdr$, as far as the vertical part of the force U is concerned. Hence the rise from this cause

$$= -\frac{1}{g}\int R dr = \frac{2\mathrm{a}\epsilon}{g}\int R\mu d\mu$$

from the equation to the ellipse. Owing to the smallness of the force, this is a quantity of the second order of small quantities, and is therefore to be rejected. Hence we need consider only the horizontal portion of the force. The rationale of this is evident. The horizontal force acts by *accumulation* in the direction of its action and piles up the water,

whereas the vertical force acts by the *difference* of vertical pressure, which it causes the successive vertical columns of water to produce.

214. We shall take some examples of horizontal forces in nature.

It has been stated (Art. 192) that although there are causes (such as the Himalayas and the Ocean) which produce a considerable amount of local attraction, yet that on the whole they very nearly balance each other. The following three examples are therefore solely for applying the formula.

Ex. 1. The Himalayas attract places along the coast of Hindostan with a force varying nearly inversely as the distance from a line running E.S.E. and W.N.W. through a point in latitude 33° and longitude 77° 42′, and equal to g tan 7″ at 1020 miles' distance: (see *Phil. Trans.* 1855, pp. 91, 94; also 1859, p. 793). Find the effect this would have upon the sea-level between Cape Comorin and Karachi, which are about 1600 and 775 miles from this line, if there were no counteracting cause, as it is believed there is (Art. 192).

In this case $U=-g\tan 7''\,(1020\div u)$; u is the distance from the line. We may take the arc for the chord. Therefore the rise of sea-level from this cause

$$\begin{aligned} &= 1020\tan 7''\log_e\frac{1600}{775}\text{ miles} = 0.0346\times\frac{0.3148}{0.414} \\ &= 0.025\text{ mile} = 132\text{ feet.}\end{aligned}$$

Ex. 2. As the distance from the line increases, the force will vary more as the inverse *square.* Suppose that to the distance 1020 miles it varies as the inverse distance, and beyond that as the inverse square. For the first we must integrate as above: thus

$$0.0346\log_e\frac{1020}{775} = 0.0346\,\frac{0.1193}{0.434} = 0.0095\text{ mile} = 50\text{ feet.}$$

For the more southern part $U=-g\tan 7''\,(1020\div u)^2$, and the rise of the level

$$=1020\tan 7''\left(\frac{1020}{1020}-\frac{1020}{1600}\right)=0.0346\times\frac{29}{80}=0.01254\text{ mile}=66\text{ feet.}$$

The sum of these is 116 feet, and is somewhat less than the result before obtained.

Ex. 3. If u be the distance, in linear degrees, of the parallel of any place on the west coast of Hindostan from that of Cape Comorin, then the force acting towards the north at any point of that coast, arising from the deficiency of matter in the Ocean, may be approximately represented by the following formula (see Art. 89, Ex. 2):

$$(0.000095556839 - 0.000002836162u + 0.000000004072u^2)\, g.$$

Hence at this place the sea-level would be higher than at Cape Comorin, in consequence of this cause, by

$$0.000095556839u - 0.000001418081u^2 + 0.000000001357u^3.$$

Karachi is about 17° north of Cape Comorin. Hence from this cause, the sea would be higher at Karachi than at Cape Comorin by 0.00122 of a linear degree = 0.8489 mile = 448 feet, if there were no other cause in operation to counteract it. This added to the result of the last example makes 564 feet for the rise at Karachi. But as it has been shown that the variations of density in the crust are very nearly complementary to the visible variations in mountains and ocean, the effect of these invisible variations very nearly counteracts that of the visible, and the change in sea-level is insensible.

Ex. 4. To find how much higher the sea-level stands on the shores of Great Britain than it would, if the Ocean in the New Zealand hemisphere were to become land, all other things remaining as at present.

If a great circle be drawn upon the earth as an equator, New Zealand and Great Britain (which are nearly in each other's antipodes) being its poles, the New Zealand hemisphere is nearly all water. We must find the effect of the deficiency of matter in this ocean hemisphere in producing horizontal local attraction in the opposite hemisphere.

(1) We will suppose that this effect is the same as if the New Zealand ocean were of the form of a hemi-spheroidal meniscus, of thickness h at New Zealand: then by Art. 90,

the horizontal attraction at a place in the Great Britain hemisphere at a distance θ° from New Zealand ($= W$)

$$= (0.1446 \sin\theta + 0.0958 \sin 2\theta + 0.0244 \sin 3\theta)\frac{h}{a}\frac{\rho}{2.75}g,$$

ρ being the deficiency of density in the ocean.

Hence, supposing that this place is connected by a canal (as is the case in the North and South Atlantic Ocean) with the New Zealand hemisphere, the consequent elevation of the sea-level there is

$$= \int_{\frac{\pi}{2}}^{\theta} \frac{W}{g}\, d\,.\, a\theta = \frac{\rho}{2.75} h\,(-0.1446 \cos\theta - 0.0479 \cos 2\theta - 0.0081 \cos 3\theta - 0.0479).$$

The density of sea-water $= 1.028$: hence $\rho = 2.75 - 1.028 = 1.72$, and the elevation of the sea-level

$$= -0.626\,h\,(0.0479 + 0.1446 \cos\theta + 0.0479 \cos 2\theta + 0.0081 \cos 3\theta)$$

$$= 0.626h\,(0.1527 - 0.0958) \text{ at Great Britain}$$

$$= 0.626 \times 0.0569h = 0.036188h \text{ mile}$$

$$= 382 \text{ feet, if } h = 2 \text{ miles.}$$

(2) Suppose that the ocean in the New Zealand hemisphere is considered to be of the form of a meniscus, the thickness at the pole being zero, and at the edge h. Then by Art. 92, the elevation of the sea-level

$$= -0.626\,(2.0608 \cos\theta + 0.9442 \cos 2\theta + 0.2454 \cos 3\theta + 0.9442)\,h$$

$$= 0.626\,(2.3062 - 1.8884)h = 0.626 \times 0.4178\,h$$

$$= 0.2616h = 2762 \text{ feet, if } h = 2 \text{ miles.}$$

(3) Suppose that the ocean is regarded as uniformly deep. Then by Art. 91 the elevation of the sea-level

$$= -0.626\,(2.2054\cos\theta + 0.9921\cos 2\theta + 0.2535\cos 3\theta + 0.9921)\,h$$

$$= \quad 0.626\,(2.4589 - 1.9842)\,h = 0.626 \times 0.4747\,h$$

$$= \quad 0.2972\,h = 1570 \text{ feet, if } h = 1 \text{ mile.}$$

The average of these three results is

$$= \frac{1}{3}(382 + 2748 + 1570) = 1567 \text{ feet.}$$

215. There is no doubt that the solid parts of the earth's crust beneath the Pacific Ocean must be denser than in the corresponding parts on the opposite side, otherwise the ocean would flow away to the other parts of the earth. (See Art. 192.) The following reasoning will explain this. Suppose the earth to be a sphere. Through any point on it suppose a surface drawn separating a thin portion on the right hand and through the same point a similar surface separating a like portion on the left. The sphere consists, then, of three parts, the middle portion being of a symmetrical form and attracting the point in the direction of the radius, and the two slender slices attracting it equally to the right and left of that radius. If one of these slices became fluid and of less density than the other, its attraction would be overcome by that of the other, and the fluid would be drawn away to the other parts of the sphere. It does not follow that the whole of the fluid would be drawn over. The above process would go on till the surface of the fluid at the circumference of the slice had become so inclined as to be at right angles to the direction of the resultant attraction of the whole mass, solid and fluid. If, however, a narrow channel were cut through this circumference (which would otherwise act as an embankment) the whole of the water would be drawn off.

Now in the case of the earth there is a channel opening a passage from the New Zealand hemisphere into the opposite one, viz. the North and South Atlantic, and yet the ocean remains in that hemisphere. There must, therefore, be some excess of matter in the solid parts of the earth between the Pacific Ocean and the earth's centre which retains the water in its place. This effect may be produced in an infinite

variety of ways; and therefore, without data, it is useless to speculate regarding the arrangement of matter which actually exists in the solid parts below.

216. Some geologists appear to consider, that the ocean in the northern hemisphere must have stood at a much higher level than it does at present, at the breaking up of the great northern glacial period which led to the phenomena of the drift. In 1865 Mr Croll of Glasgow proposed the ingenious hypothesis, that this may have been due to the presence of an enormous Ice-Sheet in the northern hemisphere, which would alter the centre of gravity of the earth and modify the level of the ocean. The ice-sheet he supposed had been deposited from the air during countless ages in snow and hail and held fast in a solid mass. His notion rests on the following hypothesis, which changes in the eccentricity of the earth's orbit appear to suggest; that in time past a grand cosmical change has been going on, according to which the northern and the southern hemispheres (at any rate the higher portions of them) have been alternately bound up in ice, and have alternately yielded to milder influences, when the ice-sheet has become broken up, moving off in huge fragments which have caused the phenomena of the drift, and has finally disappeared. For the sake of calculation he takes the ice-sheet to be a hemispherical meniscus of a certain thickness (h) at the pole and gradually getting thinner towards the equator, where he supposes it to be zero. The specific gravity of ice is 0.92 and of superficial rock 2.75. Hence the ratio of the densities of ice and rock is 1 : 3. It will be a curious and interesting exemplification of our formula to ascertain whether the attraction of such an ice-sheet would raise the ocean on which it floated to a height which the phenomena require. A practical difficulty is that the amount of water necessary to form such a sheet would be enormous. Nevertheless, as the thickest part would be in the less extensive portions, the polar, the water at present in the northern hemisphere may be sufficient to have supplied it.

Ex. 5. To find how much a hemispherical Ice-Sheet in the northern hemisphere, as described above, would raise the level of the ocean beneath it.

By Art. 90, formula (2), horizontal attraction of this meniscus of ice towards the north pole, on a point at its surface ϕ° from that pole (U)

$$= (0.1446 \sin \phi + 0.2042 \sin 2\phi + 0.0244 \sin 3\phi) \frac{h}{a} \frac{g}{3}.$$

Hence, Rise in Level of the Ocean

$$= \frac{1}{g} \int U du = -\frac{a}{g} \int U d\phi$$

$$= (0.0482 \cos \phi + 0.0340 (\cos 2\phi + 1 + 0.0027 \cos 3\phi) h,$$

integrating from the equator, where $\phi = 90^\circ$.

In the two latitudes 45° and 60°, that is, when $\phi = 45^\circ$ and 30°, this gives $0.0662h$ and $0.1428h$ feet, h being expressed in feet.

Mr Croll takes h to be 7000 feet. This gives the rise of the ocean in latitudes 45° and 60° to be 463 and 1000 feet. These heights of the ocean above its present level may have been somewhat greater, owing to the condition of the southern hemisphere at the time. For when the northern hemisphere had attained its maximum of glaciation, the southern hemisphere would have lost the ice-sheet in which it, in its turn, had been previously enveloped, and the released waters would have flowed, aided also by the attraction of the northern ice-sheet, from the southern hemisphere into the northern, and have correspondingly raised the general level of the ocean in the northern hemisphere above its previous state, and above the state to which it again came on the breaking up of the northern ice-sheet and the gradual re-formation of the ice-sheet in the south. To estimate the influence of this cause, we have but to make use of formula (1) of Art. 90, and integrate from $\theta = 0$. Then rise of the ocean from this cause

$$= -(0.1446 \cos \theta + 0.0479 \cos 2\theta + 0.0081 \cos 3\theta) h.$$

In latitudes 45° and 60°, $\theta = 135^\circ$ and 150°; and the rise of the ocean level $= 0.0322h$ and $0.0347h$ feet. Putting $h = 7000$ feet, these become 225 and 243 feet. It is evident that the southern ice-sheet is at the present time only very partially formed, and therefore cannot produce results near so large as

those in lowering the present sea-level in the northern hemisphere. Nor is it to be supposed that when the glaciation in the northern hemisphere was at its maximum, the ice-sheet reached down so far as the equator. The results obtained must therefore be somewhat diminished on this account.

But there is a circumstance well worth considering, which may have considerably increased the depth of water during the Drift Period. The thickness of the ice in the latitudes 45° and 60° would be 3500 and 5250 feet; that is, half and three-quarters of the thickness at the pole, because the thickness varies as the square of the sine of the latitude. Now, as the ice-sheet began to melt and break up into portions, the force of attraction would not change till the water from the melted ice had had time to flow away towards the open sea in the equatorial latitudes; for the water derived from the ice, though fallen down a few hundred feet into the channels and lakes below in the general mass, would attract very much as the ice itself did. And as the gradient, the slope of the ocean, would be slight, the tendency of the water to flow out, and of the sea-level to sink, as the ice was turned into water, would be feeble, and the outflow much obstructed by the windings of the channels and the counter-currents produced by impact upon the enormous ice-fields into which the ice-sheet would at first be broken. It appears, then, not improbable that the supply of water in the northern latitudes from the melting of masses between 3000 and 7000 feet high would be far more rapid than the outflow of water owing to the solid mass becoming less; and the consequence would be that during the Drift period the ocean in the northern latitudes would stand at a much higher level, for a considerable time, than the height we have calculated for latitudes 45° and 60°; and vast icebergs with huge boulders fixed in their lower parts might find ample depth of water to carry them over all the places where the phenomena of abrasion and drift-deposit have been found.

CONCLUSION.

NATURE OF THE EVIDENCE THAT THE EARTH WAS ONCE FLUID.

217. The results to which the Fluid Theory leads accord remarkably with the observed phenomena of the earth's surface; and especially so in the spheroidal form, the ratio of the axes, and the law of gravity which it gives, as noticed particularly in Arts. 110, 118, 123. If the converse of these results could be assumed to be true also, the former fluidity of the earth would be established beyond all controversy. But Professor Stokes has shown, as noticed in Art. 141, that if the form be a spheroid of equilibrium the change of gravity at the surface will vary as the difference in the squares of the sine of the latitude, and the relation between ellipticity and gravity known as Clairaut's Theorem will be true, whatever be the arrangement of the interior. Now the ocean covers a vast portion of the globe, and is of itself necessarily a surface of equilibrium; and the elevation of continents and mountains above the sea-level is but a very small fraction of the radius, such as to show that the whole surface may be approximately regarded as a surface of equilibrium; and the various trigonometrical surveys show that to an equal degree of approximation the surface is spheroidal. Hence the earth's mean surface may certainly be regarded as a spheroid of equilibrium; and therefore, so far, according to Professor Stokes' investigations, the observed law of gravity and other phenomena at the surface can teach us nothing regarding the arrangement of the interior of the mass: the arrangement according to the fluid law may be the actual arrangement, or it may be any one of an infinite variety of other arrangements. But what is the *particular* form of the spheroid which defines the earth's surface? The further the investigation

een carried the more does it appear, that of all the variety of spheroidal forms, it has that numerically form, which a fluid body of the same volume and would have if in equilibrium in itself and revolving day round a fixed axis; see Arts. 131, 173. What the earth to have this form? The Fluid Theory gives *causa* for this form. And, even supposing that were some *à priori* reason why the form should be eroid, the chances would be small in favour of its ning to be the spheroid of this precise form, irre-ely of its being brought about through the influence e *vera causa*, such as the Fluid Theory presents.

essor Stokes' view on this subject, published some ago, is as follows:

may be well to consider the degree of evidence af- by the figure of the earth in favour of the hypothesis earth's original fluidity.

the first place it is remarkable that the surface of rth is so nearly a surface of equilibrium. The eleva- f the land above the level of the sea is extremely compared with the breadth of the continents. The of the sea must of course necessarily be a surface of rium, but still it is remarkable that the sea is spread ormly over the surface of the earth. There is reason k that the depth of the sea does not exceed a very iles on the average. Were a roundish solid taken at n, and a quantity of water poured on it, and allowed le under the action of the gravitation of the solid, bability is that the depth of the water would present t of uniformity, and would be in some places very Nevertheless the circumstance that the surface of th is so nearly a surface of equilibrium might be at- d to the constant degradation of the original elevations the lapse of ages.

the second place, it is found that the surface is very an oblate spheroid, having for its axis the axis of n. That the surface should *on the whole* be protu- about the equator is nothing remarkable, because even he matter of which the earth is composed arranged trically about the centre, a surface of equilibrium

would still be protuberant in consequence of the centrifugal force; and were matter to accumulate at the equator by degradation, the ellipticity of the surface of equilibrium would be increased by the attraction of this matter. Nevertheless the ellipticity of the earth is much greater than the ellipticity ($\frac{1}{2} m$) due to the centrifugal force alone, and even greater than the ellipticity which would exist were the earth composed of a sphere touching the surface at the poles, and consisting of concentric spherical strata of equal density and of a spherico-spheroidal shell having the density of the rocks and clay at the surface [or $1 \div 420$ nearly]. This being the case, the regularity of the surface is no doubt remarkable; and this regularity is accounted for on the hypothesis of original fluidity.

"The near coincidence between the numerical values of the ellipticity of the terrestrial spheroid obtained independently from the motion of the moon, from the pendulum, by the aid of Clairaut's theorem, and from direct measures of arcs, affords no additional evidence whatever in favour of the hypothesis of original fluidity being a direct consequence of the law of universal gravitation...These results [that every equatorial axis and the axis of rotation are principal axes] would follow as a consequence of the hypothesis of original fluidity.

"The phenomena of precession and nutation introduce a new element to our consideration, namely, the moment of inertia of the earth about an equatorial axis. The observation of these phenomena enables us to determine the numerical value of the [moment of inertia] if we suppose ϵ known otherwise. Now, independently of any hypothesis as to original fluidity, it is probable that the earth consists approximately of spherical strata of equal density. Any material deviation from this arrangement could hardly fail to produce an irregularity in the variation of gravity, and consequently in the form of the surface, since we know that the surface is one of equilibrium. Hence we may assume when not directly considering the ellipticity, that the density is a function of the distance from the centre...

"Now the preceding results will not be sensibly affected by giving to the nearly spherical strata of equal density

one form or another, but the form of the surface will be materially affected. The surface in fact might not be spheroidal at all, or if spheroidal, the ellipticity might range between tolerably wide limits. But according to the hypothesis of original fluidity the surface ought to be spheroidal, and the ellipticity ought to have a certain numerical value depending upon the law of density. If then there exist a law of density, not in itself improbable *à priori*, which satisfies the required conditions respecting the mean and superficial densities, and which gives to the ellipticity and to the annual precession numerical values nearly agreeing with their observed values, we may regard this law not only as in all probability representing approximately the distribution of matter within the earth, but also as furnishing, by its accordance with observation, a certain degree of evidence in favour of the hypothesis of original fluidity. The law of density usually considered in the theory of the figure of the earth is a law of this kind."—*Cambridge and Dublin Math. Journal*, Vol. IV. p. 210.

It has been suggested, that a meteoric origin of the earth might be conceived, which would account for the surface phenomena as well as the fluid theory, and be as much a *vera causa*. According to this hypothesis the earth has grown up by the accretion of meteorolites in the course of ages, falling nearly alike on all sides and being slightly redistributed superficially by the effect of degradation, so as to keep the surface always level. The mass would be arranged in spheroidal layers of revolution, which, however, would cease to be level surfaces as the growth went on, nor would the relation between the density and ellipticity for the layers be the same as in the fluid theory. But this can hardly be put as a *vera causa*, on a par with the fluid theory: for there are various arbitrary conditions assumed which make it far less admissible. The fall of meteorolites must have been almost uniform on all sides. No cause can be assigned for this; in fact it is highly improbable. Moreover the density of the meteorolites must have been very different at different times, as the density of the interior of the earth is much greater than that of the surface. There is no assignable cause for this periodic change in the character of the meteorolites. It is invented to

support the hypothesis. Both these are arbitrary assumptions, whereas for the truth of the fluid theory all we require is, that the earth has been in former ages much hotter than at present, so far as to melt the materials sufficiently to make them amenable to the laws of fluid pressure. For this hypothesis there is every ground of presumption; and that the materials are capable of being melted even now in certain parts of the mass is shown in volcanoes and the lava which they eject.

Postscript to Articles 28—32.

Should the student desire to have in a more concise form the pooof we have given in Arts. 28—32 of the important Proposition, that every function of μ and ω, which does not become infinite between the limits, can be expanded in a series of Laplace's Functions, the following will suit his purpose. But a perusal of the more detailed proof in those Articles will probably give him a more complete insight into these singular functions.

The radius of the sphere of which C (diagram, p. 24) is the centre is unity, Q a fixed point on its surface, P a variable point, θ $(=\cos^{-1}\mu)$ and ω are spherical co-ordinates of Q, and θ' $(=\cos^{-1}\mu')$ and ω' of P, measured from A and a fixed great circle through A and a. The surface is divided into four-sided elements by great circles making the same angles, each $= d\omega'$, one with another and by planes at right angles to Aa at equal distances, each $= -d\mu'$. These elements will be of different shapes, according to their situation in the lune to which they belong; but they are all of the same area, because any one of them $= d\omega' \sin\theta' . d\theta' = -d\mu' d\omega'$, and $d\mu'$ and $d\omega'$ are the same for all the elements.

If p be the cosine of the angle which CP makes with CQ; $P_1, P_2 \ldots P_i \ldots$ Laplace's Coefficients, then

$$p = \mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos(\omega-\omega'),$$

and

$$\frac{1}{\sqrt{1+c^2-2cp}} = 1 + P_1c + P_2c^2 + \ldots + P_ic^i + \ldots\ldots$$

Suppose the surface of the sphere is divided in exactly the same way with respect to Q as origin, as with respect to A; let $\cos^{-1}p$ and ψ be the circular co-ordinates to P from Q and round Q; and let $-\delta p$ and $\delta\psi$ be the differentials of p and ψ, δ being used as distinct from d to indicate that the differentiation is with respect to a different division of the sphere. Then as before the elements of the surface are all equal to each other in area, and are the same in number. Hence

$$-d\mu' d\omega' = -\,\delta p\,\delta\psi.$$

Differentiate both sides of the above expansion with respect to c, multiply by $2c$, add, and multiply by $-d\mu' d\omega'$ or its equivalent $-\delta p\,\delta\psi$,

$$\therefore -\{1 + 3P_1 c + \ldots + (2i+1) P_i c^i + \ldots\}\, d\mu' d\omega' = \frac{-(1-c^2)\,\delta p\,\delta\psi}{(1+c^2-2cp)^{\frac{3}{2}}}$$

$$= -\frac{1+c}{c}\,\delta\psi \left\{\frac{1-c}{\sqrt{1+c^2-2c\,(p+\delta p)}} - \frac{1-c}{\sqrt{1+c^2-2cp}}\right\} \text{ ultimately.}$$

When $c = 1$ both terms within the brackets vanish, except when $p = 1$, in which case the second $= -1$. Hence

$$-\{1 + 3P_1 + \ldots + (2i+1) P_i + \ldots\}\, d\mu' d\omega' = 0,$$

except when $p = 1$ (i.e. at Q); and then it $= 2\delta\psi$.

Let $F(\mu', \omega')$ be any function of μ' and ω' which does not become infinite between the limits. At Q it becomes $F(\mu, \omega)$. Hence

$$-\{1 + 3P_1 + \ldots + (2i+1) P_i + \ldots\}\, F(\mu', \omega')\, d\mu' d\omega' = 0,$$

except at Q, when it $= 2\delta\psi\, F(\mu, \omega)$.

If we integrate the first side between the limits $\omega' = 0$, $\omega' = 2\pi$; $\theta' = 0$, $\theta' = \pi$ (that is, $\mu' = 1$, $\mu' = -1$), it is the same as adding together all elementary quantities, infinite in number, represented by the first side of the above equality. But the addition of all the corresponding quantities on the second side amounts to adding together all the elementary quantities like $2\delta\psi\, F(\mu, \omega)$ which meet in Q all around it,

which sum $= 4\pi F(\mu, \omega)$. Hence, changing the sign and the order of the integration with respect to μ,

$$F(\mu,\omega)=\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}(1+3P_1+\ldots+(2i+1)P_i+\ldots)F(\mu',\omega')\,d\omega'd\mu'.$$

From this definite integral μ' and ω' disappear, and the ith or general term is a function of μ and ω, and is a Laplace's Function, for it satisfies Laplace's Equation. Hence the Proposition is true.

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